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ABSTRACT

This study investigated the differences in learning and retention of students informed and not informed of behavioral objectives and the learning hierarchy of a unit of instruction. The subjects were 88 elementary education majors at Towson State College, Baltimore, Maryland, randomly assigned to four treatments. Self-instructional material for the unit was developed based upon the learning hierarchy. The treatment material, consisting of ten activity booklets, was administered by the experimenter for eight consecutive class days. After completion of the unit, posttests were administered to compare the degree of learning and the amount of retention. The results do not substantiate the thesis that informing students of behavioral objectives and/or the learning hierarchy can enhance their performance on an immediate achievement test. However, giving students statements and examples of behavioral objectives is an instructional method that will result in resistance to forgetting. (Author/EK)

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LEARNING AND RETENTION BY INFORMING STUDENTS OF BEHAVIORAL OBJECTIVES AND THEIR PLACE IN THE HIERARCHICAL LEARNING SEQUENCE

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November 1969

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Baltimore, Maryland

November 1969

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U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

ABSTRACT

Title of Project: Learning and Retention by Informing Students of Behavioral Objectives and their Place in the Hierarchical Learning Sequence

This study was conducted to investigate the question:

> If a group of students is informed of the behavioral objectives and the learning hierarchy of a unit of instruction and another group of students receiving the same unit of instruction is not so informed, will there be differences in effect on learning and retention?

It was expected that those students who are informed of the behavioral objectives of an activity will perform higher on achievement and retention posttests than those students who are not so informed. Moreover, it was expected that students who are informed of the activity's place in a hierarchical learning sequence (designed after Robert Gagné's cumulative learning model) in addition to being informed of the behavioral objectives of the activity will perform higher on achievement and retention posttests than those who are informed just of the behavioral objectives of the activity. The author's research was designed to determine whether for

a specific population with specific treatments data could be obtained to support the above expectations. Accordingly, eighty-eight elementary education majors in a four-year college were blocked on ability levels and randomly assigned to four treatments. While receiving different information about the behavioral objectives and the hierarchical learning sequence, all four groups received the same set of self-instructional text material covering a mathematical unit of instruction. Nine hypotheses were formulated.

The subjects were students enrolled in four sections of the second semester of a two-semester sequence mathematics course for elementary education majors at Towson State College, Baltimore, Maryland in the spring of 1969. The students were classified according to their ability levels as reflected by their grades in the first semester of the two-semester sequence course before being randomly assigned to the following four treatments:

T_1--self-instructional text material on a mathematical unit.

T_2--self-instructional text material on a mathematical unit, with statements and examples of the behavioral objectives given to the student before each activity in the unit.

T_3--self-instructional text material on a mathematical unit with a copy of the learning hierarchy with examples of each cell in the hierarchy given to the student at the beginning and at the end of the unit.

T_4--self-instructional text material on a mathematical unit with the students given:
- (a) copies of the learning hierarchy (with cell examples) at the beginning of the unit,
- (b) statements and examples of the behavioral objectives of each activity at the beginning of each activity and a description of that activity's place in the hierarchical learning sequence.

The self-instructional material for the instructional unit was developed for this study and was based upon the learning hierarchy constructed for the experiment. The treatment material, consisting of ten activity booklets, was administered by the experimentor for eight consecutive class days. After the completion of the instructional unit, posttests were administered immediately to compare the degree of learning, and, after two weeks, to compare the amount of retention.

Through treatments-by-levels analysis of variance, the relative effectiveness of the four treatments on student achievement as measured by the immediate posttest was determined. Differential effects of the four treatments on learning the intellectual skill of the terminal task of the learning hierarchy were not significant at the 0.05 level. The treatments-by-levels interaction effects of the four treatments were also found not significant at the 0.05 level.

At a 0.05 level of significance, a repeated measures

analysis detected no differential effects, resulting from the four treatments, on student over-all performance. However, the four treatments resulted in significantly different rates of forgetting. The group of students who were given statements of each activity's objective showed a positive gain in performance over time. In addition, the differences in the over-all performance scores resulting from the four treatments were not identically reflected at each ability level.

The results of the study do not substantiate the thesis that informing students of the behavioral objectives and/or learning hierarchy can enhance their performance on an immediate achievement test. However, the study does suggest that giving students statements and examples of the behavioral objectives is an instructional method that will result in resistance to forgetting.

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This study, which was reported in the author's Ph.D. dissertation at the University of Maryland, was made possible through the efforts and assistance of several persons.

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Mrs. Beverly Gladd typed the dissertation and the materials for the experiment. Her willingness to work under the pressure of very tight time schedules enabled the study to be completed on schedule. Many friends and colleagues have assisted with expressions of thoughtfulness or contributions of time and services.

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CHAPTER I

INTRODUCTION

In 1965 Mayor, Henkleman, and Walbesser (1) stated in reference to the new mathematics and science programs in secondary schools that, while the decade 1955-65 had been one of curriculum innovation, the next should become one of research in learning and teaching of mathematics and science. The need and interest in classroom research for the benefit of curriculum reform has been expressed in other areas besides the area of teaching mathematics and science.

A principal goal of the curriculum-reform movement is increased learning and retention. There have been many research studies conducted to determine the effects of various manipulative variables on learning and retention. Many of these studies have contrasted the effect on achievement and retention of different classroom materials and different methods of instructing the student in the classroom. Although such efforts have been extensive, in a recent journal article entitled "On the Assessment of Retention Effects in Educational Experiments" Kenneth H. Wodtke (2) of The Pennsylvania State University made a plea for an even greater emphasis on long-term follow-up measures in studies of the effects of instructional treatments. He argued that there is a need for the investigation of instructional treatments specifically designed to facilitate such long-term effects. He pointed out that, even though some instructional variations might be of little value in facilitating the amount of learning, these same variations might have their primary effects on long-term retention. That is, an instructional treatment might produce relatively inefficient learning, but greater resistance to forgetting than some other treatments.

Advocates of behavioral objectives for education, such as Gagné (3), Mager (4), and Walbesser (5) have called for more specific statements of purpose and expected outcomes in new curriculum development. The American Association for the Advancement of Science has developed a curriculum entitled Science--A Process Approach (6), in which the objectives of the curriculum are stated in terms of what the student is to do rather than in terms of verbalizable knowledge that the student is to know. Behavioral descriptions of the objectives of curriculum has become basic to some new proposals for curriculum revision and development. Another dimension of curriculum design which has begun to play an important role in new curriculum developments is the construction of learning hierarchies. Gagné has hypothesized that intellectual skills that are learned

> . . . have an ordered relation to each other, such that subordinate ones contribute positive transfer to superordinate ones [7].

Gagné refers to learning sequences which exhibit such ordered relations between the behavioral objectives as learning hierarchies. There have been several recent researches reported which investigated the problems of hierarchy construction and behavioral description of learning outcomes. Among the recent contributions to this literature are those by Gagné (8, 9), Walbesser (10, 11, 12), Walbesser and Carter (13), and Engel (14).

It is in the context of these three points (1) need for long-term studies, (2) need for behavioral objectives, and (3) need for learning hierarchies, that led to the posing of the following question: If a group of students is informed of the behavioral objectives and the learning hierarchy of a unit of instruction and another group of students receiving the same unit of instruction is not so informed, will there be differences in effect on learning and retention?

A review of the literature in Chapter II seeks to establish the rationale for the question. Chapter II also includes statements of the research hypotheses tested in this study which are associated with the question.

Chapter III describes the procedure followed in the research. The findings of the investigation are presented in Chapter IV. Chapter V presents the conclusions of the study based upon the findings.

FOOTNOTES--CHAPTER I

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[14] Roberta S. Engel, "An Experimental Study of the Effects of Stated Behavioral Objectives on Achievement in a Unit of Instruction on Negative and Rational Base Systems of Numerations" (unpublished Master's thesis, The University of Maryland, College Park, 1968).

CHAPTER II

REVIEW OF LITERATURE AND RELATED RESEARCH

The first section of this chapter discusses the application of behavioral objectives to curriculum design. The utilization of learning hierarchies in designing curriculum is discussed in the second section. The last section is concerned with the importance of behavioral objectives and learning hierarchies from the point of view of student acquisition.

BEHAVIORAL OBJECTIVES AND CURRICULUM DESIGN

The expression "behavioral objectives" is in vogue in current educational circles. In a description of how fashionable the topic of behavioral objectives is today in the field of education, Walbesser stated in his text, _Constructing Behavioral Objectives_:

> Few professional meetings of educators escape from talk about objectives. Individuals involved with the public and private funding agencies speak of the need for more specific statements of purpose and expected outcomes. Aspiring developers of

> instructional systems identify behavioral descriptions as fundamental to their efforts. Product development based upon behavioral specification is common in the descriptions of instructional materials promised by the new industry based educational films [1].

Supported by such contributions to the field as Mager's (2) programmed text on preparing behavioral objectives and Gagné's (3) volume on learning, curriculum developers who affirm the value of behavioral objectives seek to have goals of education stated in behavioral terms so that the instructor will know what the students are expected to be able to do after they have had the learning experiences. The Maryland Elementary Mathematics Program (MEMIP) exemplifies a curriculum project that is committed to reliably observable human behavior as the basis for development of instructional materials (4, 5). One of MEMIP's stated goals is to demonstrate the selection of inservice materials for an instruction program based upon a set of behavioral objectives (6).

The idea of specifiying performance objectives in education did not originate in the 1960's. Bobbitt wrote in 1918:

> Human life, however varied, consists in its performance of specific activities. Education that prepares for life is one that prepares definitely and adequately for these specific activities. However numerous and diverse they may be for any social class, they can be discovered. This requires that one go out into the world of affairs and discover the particulars of which these affairs

> consist. These will show the abilities, habits, appreciations, and forms of knowledge that men need. These will be the objectives of the curriculum. They will be numerous, definite, and particularized. The curriculum will then be that series of experiences which childhood and youth must have by way of attaining those objectives [7].

Ralph W. Tyler and his associates were concerned in 1934 with the objectives of education and their relationship to the measurement of achievement. In a 1934 article, Tyler made the following comment on characteristics of educational objectives which he considered desirable for the purposes of evaluation:

> Each objective must be defined in terms which clarify the kind of behavior which the course should help to develop among the students; that is to say, a statement is needed which explains the meaning of the objectives by describing the reactions we can expect of persons who have reached the objective. This helps to make clear how one can tell when the objective is being attained since those who are reaching the objective will be characterized by the behavior specified in this analysis [8].

By the second half of this century, many individuals placed great emphasis on the need for specifying educational goals in terms of what students will be able to do after the learning experience (9, 10, 11). The Mid-Century Committee on Outcomes in Elementary Education, sponsored by the United States Office of Education, Educational Testing Service, the Russell Sage Foundation, and the Department of Elementary School Principals of the National Education Association, attempted to identify desirable objectives of elementary education (12). A similar effort reported by French (13) was made for secondary schools. Bloom's _Taxonomy of Educational Objectives_ was another effort to define intellectual abilities in behavioral terminology. Bloom stated in a description of his effort to produce a classification of educational outcomes:

> What we are classifying is the intended behavior of students--the ways in which individuals are to act, think, or feel as the result of participating in some unit of instruction [14].

A major source of research in curriculum design that has emphasized specifying precise behavioral curriculum objectives is the research of the military services. Gagné (15) summarized an experimental study of the training of electronic maintenance personnel that was reported by French. Forty graduates of a course in the mechanics of an airborne bombing navigational system (called the K-system) were given additional instruction in tracing the flow of information through the system as exemplified in a number of equipment problems. One-half of the group received this instruction on an actual system while the other half received the instruction on a training device called the MAC trainer. The additional instruction was designed to help the students acquire the behaviors needed for diagnosing malfunctions of the

K-system. An analysis of the objectives prior to the experiment had revealed that the objective of successfully performing troubleshooting on the K-system was not adequately represented in the regular course of instruction. Results of the experiment indicated that the instruction which was focused upon specific behavioral objectives was effective in improving the performance of the mechanics.

Two studies related by Smith (16, 17) reported the effectiveness of specifying objectives for training purposes that are based on job analysis. A study of the required performance of the rifleman in combat resulted in the development of the TRAINFIRE program. The study revealed that among the rifleman's major difficulties in combat were the tasks of locating and identifying targets. The old method of training the riflemen provided no practice in identifying targets. The TRAINFIRE range, established as a result of the study, enabled the soldier to obtain specific training in target detection. The second study related by Smith revealed a savings in instruction and learning time when a course for ordinance fire control technicians was based upon the specific behaviors that the technicians were expected to perform on the job. The study found that graduates of a four hundred hour course based upon training for specific job behaviors were as proficient in electronic troubleshooting as graduates of the regular thousand hour course. An analysis of the job the soldier was expected to perform yielded the behavioral objectives for the experimental training in each of the two studies mentioned. Commenting on the basic source of training objectives, Smith stated:

> The performance required by the soldier in his job is the basic source of training objectives. The key question is, "What must this soldier be able to do in order to do his job well?" It is highly important that the soldier be taught only the things he needs for doing his job; teaching him things that are irrelevant to the job or teaching the wrong things can be very costly [18].

The training research conducted by the armed services has implications for education. Educators have recognized the value of clearly defined objectives for the classroom teacher as well as for curriculum developers. Gagné (19) has argued that a decision needs to be made about the nature of the change in behavior sought before a decision is made concerning the kind of learning situation needed to bring about a particular change in behavior. Lindvall argued for curriculum developers to define their goals in terms of what the student will be able to do at the end of an instructional activity:

> Although curriculum plans must typically describe classroom activities that will be carried out by teachers and pupils and must describe the subject matter content that provides the basis for lessons, the plans must include more than this. If such plans are to provide the type of direction that

> will give the greatest assurance that there will be changes in what pupils learn, they should include statements that tell what the pupils will be able to do after they have had the suggested learning experiences [20].

Lindvall's argument was supported by Kurtz, a professor of botany at the University of Arizona, when he provided anecdotal evidence of the effectiveness of behavioral objectives in the classroom situation:

> But in my own classroom experience, I have found that students acquire as many, and perhaps even more, facts when the course is designed around behavioral objectives than when factual content is emphasized. Because the students feel they are "doing" and "accomplishing" things when behavioral objectives are set, the number of facts learned per student appears to increase considerably along with the acquired behaviors. Apparently students receive satisfaction from doing science rather than telling about what science has already done, and this satisfaction somehow stimulates the acquisition of more facts [21].

The literature in support of behavioral objectives seems to suggest that a major benefit to be derived from stating educational objectives in behavioral terminology is that the type of learning that is to be undertaken by the student, and the required conditions of learning, are greatly clarified. Gagné has argued that decisions concerning the type of instruction to be employed in attaining stated behavioral objectives

> . . . can usually not be made for an entity of learning as large as a "topic." Instead, they need to be made for each of the individual

> learning acts that collectively make up a topic, arranged in a hierarchical manner [22].

The next section discusses the role of such learning hierarchies in curriculum design.

LEARNING HIERARCHIES AND CURRICULUM DESIGN

In a report of a study of mathematics learning in 1962 Gagné (23) used the term "learning hierarchy" to refer to a set of specified intellectual capabilities having, theoretically, an ordered relationship to each other. Gagné hypothesized that tasks to be learned can be analyzed into a hierarchy of subordinate learning tasks which are related to each other in the psychological sense that the learning of some is prerequisite to the learning of others. Gagné found in a learning hierarchy for tasks on the addition of integers that there was positive vertical transfer from one subordinate task to another. The findings of the study revealed that success in achievement of the final task of the hierarchy was highly correlated (.87) with the number of subordinate tasks which were acquired. Other studies by Gagné (24, 25) verified the finding that the learning of higher-level intellectual skills was dependent upon the previous mastery of prerequisite lower-level intellectual skills. In a presidential address at the 1968 conference of the American Psychological Association Gagné emphasized the behavioral characteristic of the tasks which make up a learning hierarchy:

> The question is, what exactly are these entities, sometimes called capabilities, that make up a learning hierarchy? The answer I would now give is the following. They are intellectual skills, which some writers would perhaps call cognitive strategies. What they are not is just as important. They are not entities of verbalizable knowledge. I have found that when deriving them, one must carefully record statements of "what the individual can do," and just as carefully avoid statements about "what the individual knows" [26].

A study by Gagné and Bassler (27) found support for the hypothesis that such learning hierarchies as referred to in the previous paragraph aided retention of learned intellectual capabilities. The study measured immediate achievement of initially learned intellectual skills (or behavior) in non-metric geometry and retention after a nine-week interval. This study, while investigating other variables on the learning process, presented the topics in a sequence based on a hierarchy of learning sets in which acquisition of subordinate sets were required in order to learn the behavior on the next level of the hierarchy. Gagné and Bassler found that after exposure to a carefully constructed instructional program the skill acquired by a student to perform the terminal task was highly resistant to forgetting.

Utilization of the learning hierarchical approach to instruction has been made in the development of an elementary science curriculum called Science--A Process Approach. This curriculum for children in kindergarten and grades one through six, has been developed by the Commission on Science Education of the American Association for the Advancement of Science. Specific references to the learning hierarchies of Science--A Process Approach have been published by Walbesser (28), Walbesser and Carter (29), the Commission on Science Education Newsletter (30), and the Xerox Corporation (31). Gagné, discussing the utilization of learning hierarchies in the curriculum, said:

> The behavioral hierarchies constitute the "skeleton" of Science--A Process Approach and the rationale for selecting and ordering the sequence of exercises. Thus the behavioral hierarchies orient the teacher to the purposes of the program, or of any portion of it. The teacher may examine the progression of behavioral development depicted in these hierarchies, and derive from them a view of where teaching starts and where it is expected to go [32].

The Maryland Elementary Mathematics Inservice Program (MEMIP), mentioned earlier in reference to its stated goal to demonstrate the selection of inservice materials for the instructional program based upon the set of behavioral objectives, also is committed to use of learning hierarchies as an effective basis for curriculum design. The following stated objectives of MEMIP (33) exemplify the program's commitment to learning hierarchies as well as behavioral objectives:

1. Develop a set of behavioral objectives and behavioral hierarchies for an inservice instructional program in mathematics for elementary school teachers.

2. Demonstrate the selection of inservice materials for the instructional program based upon the set of behavioral objectives.

3. Demonstrate that the instructional program can effectively be ordered on the basis of the learning sequences determined by the behavioral hierarchies.

4. Demonstrate the acquisition of these behaviors by the
    (1) Inservice Leaders--Participants in inservice instructional program in mathematics held at the University of Maryland.

    (2) Local Teachers--Participants in local inservice instructional programs in mathematics directed by the inservice leaders.

On the basis of the experimental research by Gagné and his co-workers, the developmental research by the Commission on Science Education of the American Association for the Advancement of Science, and the developmental research by the Maryland Elementary Mathematics Inservice Program, the hierarchical arrangement of learning sets appears to aid the learning of intellectual skills.

BEHAVIORAL OBJECTIVES, LEARNING HIERARCHIES AND THE LEARNER

The literature reported in this chapter has been primarily focused upon the need of the curriculum designer and the teacher to know the desired expected outcome of the instruction in terms of behavioral objectives. What would be the effect on the learner if he were informed of the behavioral objectives? According to Gagné, telling the learner what is to be his performance when he has learned his lessons is a function performed by directions that seem to be very important to the learning process. Gagné (34) has also hypothesized that instructions in a learning situation fulfill the following functions:

1. They inform the learner of the performance that is expected of him.
2. They stimulate recall of subordinate knowledge.
3. They guide the learner's thinking.

He has argued that the conditions of the learning situation are not satisfied unless there is

> . . . some instruction, which includes the steps of informing the learner about the expected form of the performance expected, encouraging recall, and cuing the proper sequence of acts . . . [35].

Mager observed that:

> An additional advantage of clearly defined objectives is that the student is provided the means

> to evaluate his own progress at any place along the route of instruction, and is able to organize his efforts into relevant activities. With clear objectives in view, the student knows which activities on his part are relevant to his success, and it is no longer necessary for him to "psych out" the instructor [36].

Tyler had this to say about the importance of behavioral objectives from the point of view of the student:

> When the objectives are clearly defined and understood by the student, he can perceive what he is trying to learn [37].

Otherwise, he continued, a student must do what a survey of 100 students at the upper elementary and secondary school levels revealed. Almost all the students stated that they found out what they were to learn from three different sources: the textbooks and workbooks, what the teacher did in class, and the advice of other students.

Gagné (38) stated in 1964 that there was not much formal evidence to be found for the prediction that the probability of the learner attaining a solution will be reduced if instructions do not enable him to identify the desired terminal performance. In 1967, Smith (39) conducted experimental research in which he gave printed instructions concerning the expected outcome of each mathematics lesson to one group of slow learners while another group of slow learners was not given this instruction. Smith found no significant difference in the performance of the two groups on an achievement test. In 1968, Engel (40) conducted a study that found data to support the hypothesis that students will achieve higher on an achievement test if they are told in advance what are the objectives of each activity. She used material that was not organized around a learning hierarchy. Also, she found data to support the hypothesis that students will achieve higher on a retention test if they are told in advance the objectives of each activity. Because Engel did not use the repeated measures analysis appropriate for studying retention effects (41) her findings may not have been conclusive in terms of retention benefits.

SIGNIFICANCE OF THE INVESTIGATION

As previously mentioned, the study is investigating the effect on learning and retention of informing students of the behavioral objectives and the learning sequence of a unit of instruction.

The results of the research by Gagné and his associates, by the Commission on Science Education of the American Association for the Advancement of Science, and related research conducted by master's candidates at the University of Maryland suggest that the following question should be investigated:

> If a group of students is informed of the behavioral objectives and the learning hierarchy of a unit of instruction and another group of students receiving the same unit of instruction is not so informed, will there be differences in effect on learning and retention?

Authorities claim that teachers and curriculum developers need to identify the behavioral objectives of instruction in the framework of a learning hierarchy. Advocates such as Gagné and Walbesser have hypothesized that such identification aids the instructor to not only know what learned behavior he wishes the student to acquire, but also to know in what hierarchical sequence or sequences each subordinate behavior may be taught and learned in order for the next higher level behavior to be acquired by the student. There may be more than one learning hierarchy for teaching a particular performance; the claim is that a learning hierarchy should be used. In addition, the advocates of behavioral objectives claim that since the instructor has identified the learned behaviors he wishes the student to acquire, he then is able to evaluate the success of the instruction by whether or not the learner has acquired the desired behaviors.

After completing her study, Roberta Engel made this observation:

> . . . at no point was the learner ever made aware, by means of direct observation, of the learning sequence. One might question whether differences in achievement exist between learners who are

> informed of the objectives of the lessons accompanied by the learning sequence and those who are informed of the objectives of the lessons only [42].

The study conducted by the author differs from those reported in the literature by the point of emphasis. While utilizing the results of earlier research, the author's research shifts the emphasis from the benefits to be derived from the instructor knowing the objectives and the learning hierarchy to the benefits to be derived from informing the student of the behavioral objectives and the learning hierarchy. This research seeks to establish the benefit, in terms of learning and retention, of informing the student of the behavioral objectives of the instruction and of the dependent relationship the subordinate objectives have to each other and to the terminal objective.

HYPOTHESES TO BE TESTED IN THE STUDY

From the literature review it seems reasonable to expect that those students who are informed of the behavioral objectives of an activity will perform higher in achievement and retention posttests than those students who are not so informed (43). Moreover, it appears likely that students who are informed of the activity's place in the hierarchical learning sequence in addition to being informed of the behavioral objectives of the activity will perform higher on achievement and retention posttests than those who are informed just of the behavioral objectives of the activity.

This research study is designed to determine whether for a specific population with specific treatments data can be obtained to support the above expectations. Accordingly, elementary majors in a four-year college were randomly assigned to the four treatments (including a control treatment) delineated below. While receiving different information about the behavioral objectives and the hierarchical learning sequence, all four groups received the same set of self-instructional text material covering a mathematical unit of instruction. The first treatment listed, T_1, is the control treatment.

The four treatments are defined as follows:

T_1--self-instructional text material on a mathematical unit.

T_2--self-instructional text material on a mathematical unit with the objectives given before each activity in the unit.

T_3--self-instructional text material on a mathematical unit with students informed of the learning hierarchy at the beginning and at the end of the unit.

T_4--self-instructional text material on a mathematical unit with students informed:
(a) at the beginning of the unit of the learning hierarchy for the instructional unit, and

(b) at the beginning of each activity of the objectives of that activity and of that activity's place in the hierarchical learning sequence.

The research hypotheses associated with the question in the previous section and which reflect the stated expectations of this section are:

> Research Hypothesis 1: Giving students statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2) results in higher achievement scores for the students so informed than those students who are given no information beyond the actual instruction unit activity (T_1).

> Research Hypothesis 2: Giving students copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3) results in higher achievement scores for the students so informed than for those students who are not given any information beyond the actual instructional unit activity (T_1).

> Research Hypothesis 3: Giving students statements of the behavioral objectives (with examples) and copies of the learning sequence before each activity of an instructional unit (T_4) results in higher achievement scores for the students so informed than for those who are not given any information beyond the actual instructional unit activity (T_1).

> Research Hypothesis 4: Giving students statements of the behavioral objectives (with examples) and copies of the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2).

Research Hypothesis 5: Giving students statements of the behavioral objectives (with examples) and the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3).

Research Hypothesis 6: The differences in achievement scores resulting from the four treatments are not identically reflected at each ability level.

Research Hypothesis 7: The four treatments have differential effects on the over-all performance of students.

Research Hypothesis 8: The four treatments result in different rates of forgetting.

Research Hypothesis 9: The differences in over-all performance scores resulting from the four treatments are not identically reflected at each ability level.

DEFINITION OF TERMS

For the purposes of this study these definitions are provided:

Manipulative variables. The written information given the students informing them of the behavioral objectives and the learning hierarchy of the unit of instruction; the time interval between the completion of the instructional unit and the administration of the posttests.

Responding variable. The scores obtained by the students on the mathematics performance test.

Inform. The procedure of giving the student written statements of the behavioral objectives with appropriate examples of the tasks and/or giving to the student a printed copy of the learning sequence with or without a written description of the place in the sequence of the objective of each activity in the unit of instruction.

Instructional unit. A collection of activities designed to assist the learner to acquire the stated subordinate and terminal behavioral objectives.

Behavioral objective. The objective of an activity expressed in terms of the behavior the student is to acquire.

Terminal behavior. The behavior the learner is expected to be able to exhibit after some specified instruction and for which there are one or more behaviors the learner must acquire as prerequisites and for which there does not exist in the particular learning hierarchy a superordinate behavior.

Hypothesis of learning dependency. A collection consisting of one terminal behavior and one or more immediate subordinate behaviors.

Learning hierarchy. A cumulative learning sequence made up of hypotheses of learning dependency.

Retention. The number of learned behaviors that are remembered over a specified time interval as determined by scores of performance measures.

Rate of forgetting. The change over time in the scores made on the immediate posttest and the scores made on the same posttest administered after a delay of two weeks.

Over-all performance. The score obtained when the scores made on the two posttests by a student in the experiment are summed.

Identify. The students select (by shading or plotting on a sheet of paper) the correct set of points on a number line or on a Cartesian plane.

Name. The student supplies the correct identifying word or symbol (in written form) for a set of points on a number line or on a Cartesian plane.

Construct. The student generates a drawing which identifies a designated set of conditions.

Compute. The student determines or ascertains by mathematical means (e.g., Given $x = 2$, compute y when $y = x + 3$).

FOOTNOTES--CHAPTER II

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CHAPTER III

RESEARCH PROCEDURE

This chapter presents a detailed description of the five phases of the research procedure. The first section deals with the experimental design. The second section describes the development of the treatment materials and the criterion instruments. The third section covers procedures used in selection and assignment of subjects. The experimental procedures are detailed in the fourth section and the final section describes the statistical designs employed in analysis of the data.

EXPERIMENTAL DESIGN

Two statistical designs, treatments-by-levels analysis of variance and repeated measures analysis, were used in this study. The accompanying experimental design was of the following form:

$$\begin{array}{lllll} N_1 = 22 & R^* & T_1^{**} & O_1 & O_2 \\ N_2 = 22 & R^* & T_2 & O_1 & O_2 \\ N_3 = 22 & R^* & T_3 & O_1 & O_2 \end{array}$$

$N_4 = 22$ R* T_4 O_1 O_2

*The students were blocked first on levels of ability and then were randomly assigned to the four treatments.

**Control treatment.

This paradigm indicates that (a) the students were classified initially into levels of ability; (b) after they were classified into levels, the 88 students were randomly assigned to four groups; (c) one group was identified as the control group and received treatment T_1; (d) the second group received treatment T_2, a third group received treatment T_3, and a fourth group received treatment T_4; (e) an immediate posttest was administered providing observations O_1; and (f) a delayed posttest provided observations O_2.

This experimental design was selected because:

1. the design eliminated a pretest, hence the threat of the pretest being a confounding factor for the experiment was avoided;
2. the prerequisite of random assignment would be satisfied; and
3. the precision of the experiment would be increased by blocking on levels of ability.

INSTRUMENTATION

The two types of instruments used in this study were instructional instruments and criterion instruments. The instructional instruments consisted of the treatment materials for the experiment and included:

1. self-instructional text material for a unit of instruction in mathematics,
2. a learning hierarchy for the unit of instruction,
3. written statements of the behavioral objectives of each activity in the instructional unit, and
4. written descriptions of the place in the learning hierarchy of each activity in the unit of instruction.

These materials were constructed specifically for this study. The criterion instruments included a performance test designed to measure the ability of the students to exhibit the terminal behaviors of the learning hierarchy. The performance test was used as the immediate posttest and as the delayed posttest. In addition, quizzes were constructed and administered to the students after completion of each activity of the instructional unit. The first part of this section describes the procedure used to construct the treatment materials. The second part of the section discusses the construction of the criterion instruments.

Treatment Materials

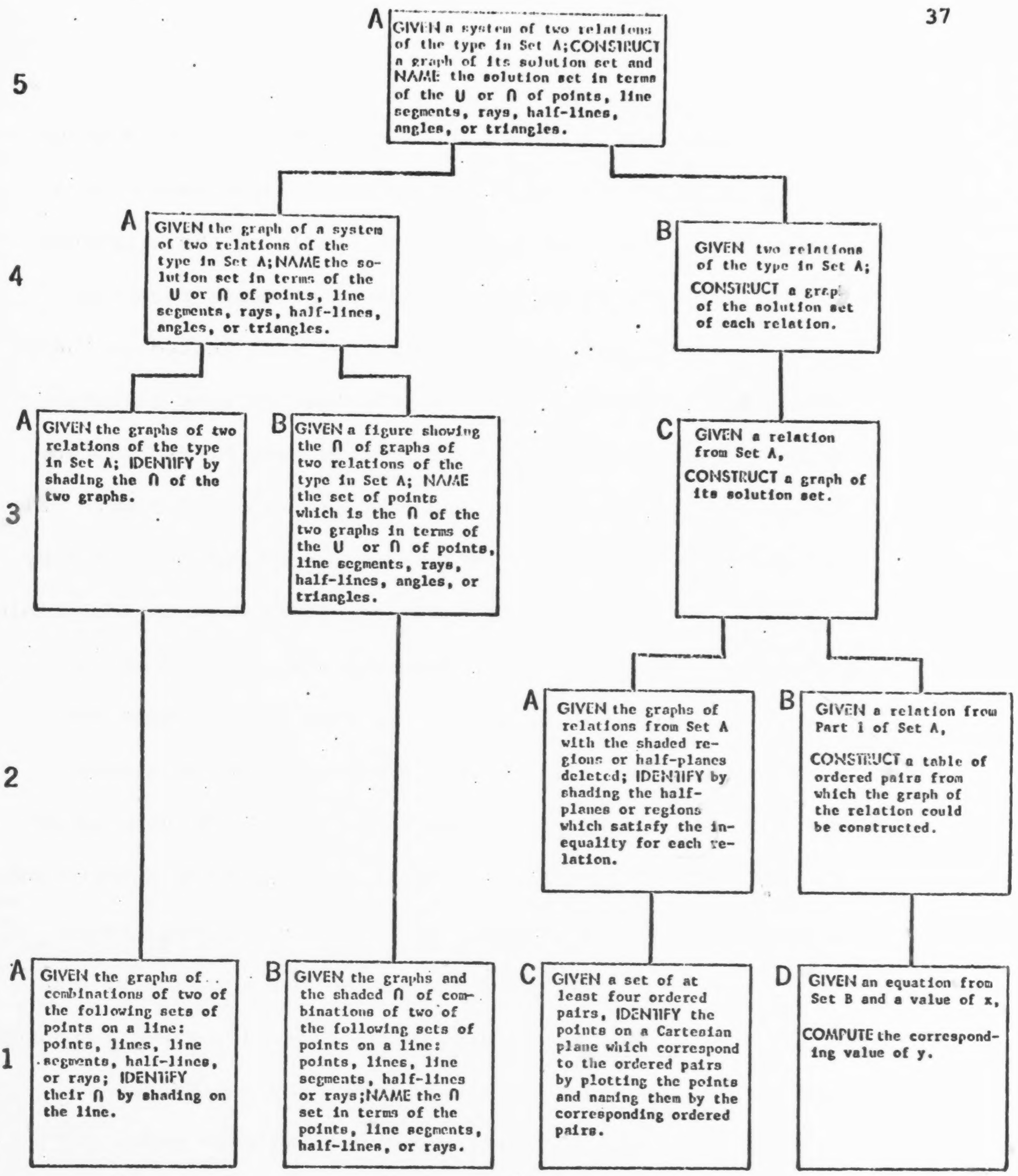
Construction of the learning hierarchy. A learning hierarchy consisting of the terminal behavior, its identified subordinate behaviors, and the hypothesized dependencies among these behaviors was constructed as the first step in the development of the treatment materials. The terminal behavior of the hierarchy stated below reflects a topic which is part of the Math 205 syllabus at Towson State College. It is written in terms of what the student is to be able to do at the end of the mathematical instructional unit:

> GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

The complete learning hierarchy, named Learning Sequence for the benefit of the students, is shown in Figure 1 on page 37.

The construction and validation of the learning hierarchy included twelve modifications of the hierarchy initially proposed. The majority of the modifications were the result of reappraisals of the terminal objective of the unit of instruction. Each time the terminal objective was redefined in an attempt to (a) adhere to the requirements of the syllabus of the college mathematics course (of which the unit of instruction was a part), and (b) adhere to the three-week period available for the instruction, several modifications of the learning hierarchy were required.

LEARNING SEQUENCE



GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

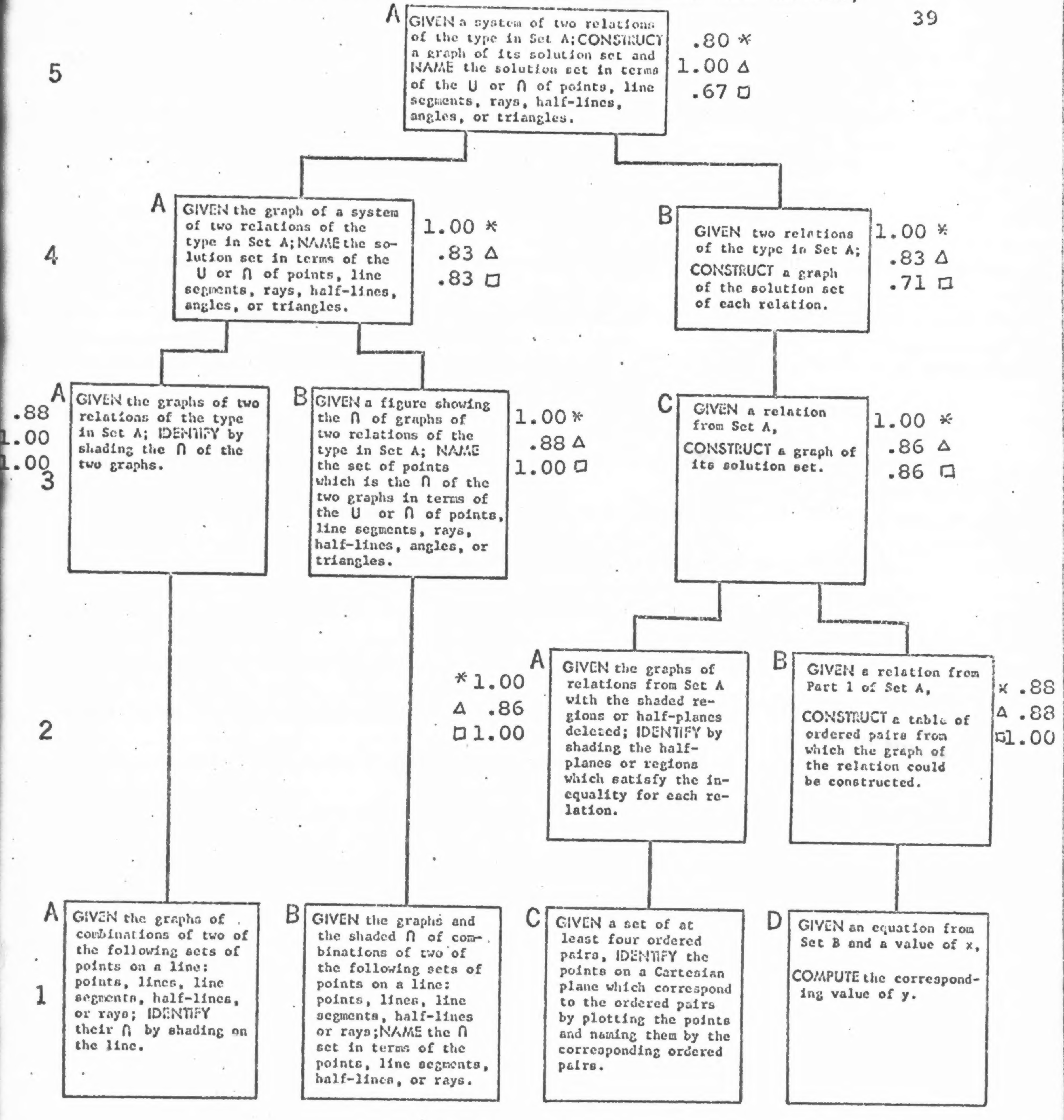
GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert ,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$
Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

FIGURE 1

Upon completion of the ninth modification, a group of eleven University of Maryland students was taught in a conventional classroom setting the skills of the hierarchy during a one-day session. The eleven students received three hours of instruction before they were tested on their ability to perform each task in the then current hierarchy. After the group of eleven students were taught and tested, the consistency, adequacy, and completeness ratios were calculated according to the validation procedures described by Walbesser (1). A value of 0.80 was selected as an acceptable level for all ratios. The hierarchy was modified in an effort to improve it at the points that did not meet the 0.80 criteria for all ratios. A second group of eleven students was taught and tested using this tenth version of the hierarchy. This procedure was continued for a third and a fourth group of students. The third and fourth groups were taught for four hours and tested for a period of two hours. After the fourth group had received instruction based upon the twelfth version of the hierarchy and had been tested, the hierarchy was found to be valid in terms of the consistency, adequacy, and completeness ratios except for two completeness ratios which fell below the 0.80 criterion. The ratios are shown on the learning hierarchy in Figure 2 on page 39. The two completeness ratios may have

LEARNING SEQUENCE
(CONSISTENCY, ADEQUACY, AND COMPLETENESS RATIOS)



5 A: GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4 A: GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4 B: GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3 A: GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the $\cap$ of the two graphs.

3 B: GIVEN a figure showing the $\cap$ of graphs of two relations of the type in Set A; NAME the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

3 C: GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2 A: GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

2 B: GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1 A: GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their $\cap$ by shading on the line.

1 B: GIVEN the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the $\cap$ set in terms of the points, line segments, half-lines, or rays.

1 C: GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

1 D: GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x+a \rvert,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x+b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c-x,\ x \text{ reals}\}$	$y = \lvert x+a \rvert$ $y = x+b$ $y = c-x$
Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

Ratio Legend:
Consistency *
Adequacy △
Completeness □

FIGURE 2

been low because of the relatively short time (four hours) available to the students to learn the behaviors of the hierarchy. Since the behaviors in the hierarchy were to be taught during eight consecutive class days over a period of three weeks, a decision was made to use the hierarchy in the form shown in Figure 1 on page 37. Recognizing that a hierarchy which is valid under one teaching situation may not be valid under a different teaching procedure, a decision was made to determine the validity of the hierarchy in the instructional setting of the experiment.

In order to obtain groups of students which appeared to be homogeneous with the sample of students subjected to the treatments of the experiment, the investigator paid each student in the four groups which were used to validate the hierarchy a wage for his time in the project. The students used for the hierarchy validation were enrolled in Mathematics 31 at the University of Maryland. Math 31 is the second course of a two-semester sequence of a college mathematics course designed for elementary education majors. As mentioned in the section entitled Subjects (page 52), the students to be used in the experimental treatments were enrolled in a similar course at Towson State College. On this basis, the groups of students used to validate the hierarchy were considered characteristic of the sample of students used in the treatments of the experiment.

Construction of the instruction unit. The self-instructional text material for the instructional unit for this experiment was written by the investigator. The text material was designed and written to teach the behaviors shown in the learning hierarchy in Figure 1 on page 37. No additional content instruction was made available to the students in class or outside of class. The material contained no homework assignments and was designed to be used only in the classroom.

All the students in the four treatment groups received the same self-instructional text material. For brevity, the self-instructional text material will henceforth be referred to as the text material. The students in the control group T_1 received only the text material. The students used in the other three treatments, T_2, T_3, and T_4, received the text material and additional information about the behavioral objectives and the learning hierarchy.

A copy of the text material given to all the students in the experiment is shown in Appendix A. Each of the eight activities comprising the unit of instruction was bound in separate booklets. While the students in treatment T_1 received only the text material, the students in treatment

T_2 received, in addition to the text material, the following:

1. A written statement of the behavioral objective of each activity in the instructional unit,

and 2. A written example of the type of task the students were expected to be able to perform at the end of each activity.

Statements and examples of the behavioral objectives of each activity were given to the students in treatment T_2 at the beginning of each activity as part of the self-instructional booklet. The first page of the self-instructional booklet for activity III is shown as Figure 3 on page 43 as an illustration of the type of statements provided the students in treatment T_2. A copy of the set of material given to the students in treatment T_2 is included in Appendix B. Because all students received the same text material, a copy of the text material is included only in Appendix A. A page insert is provided in Appendix B at the points where the text material was provided in the activity booklets.

As previously mentioned, the students in treatment T_3 received the same text material as the other students in the experiment. In addition, the treatment T_3 students were given a copy of the learning sequence with examples of each task at the beginning and at the end of the unit of instruction. The following statements were taken from one of the

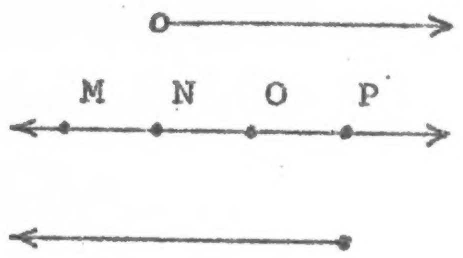
Watch Out For the Objective of Lesson III!

The goal or the objective of Lesson III is to teach you to perform the following task:

> GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays;
>
> IDENTIFY their $\cap$ by shading on the line.

Example of the type of task:

> GIVEN line $\overleftrightarrow{MP}$. One set of points on line $\overleftrightarrow{MP}$ is represented above $\overleftrightarrow{MP}$ while a second set of points on $\overleftrightarrow{MP}$ is represented below $\overleftrightarrow{MP}$.



> IDENTIFY the $\cap$ of the two sets of points by shading line $\overleftrightarrow{MP}$ in red over the appropriate portion.

Solution:

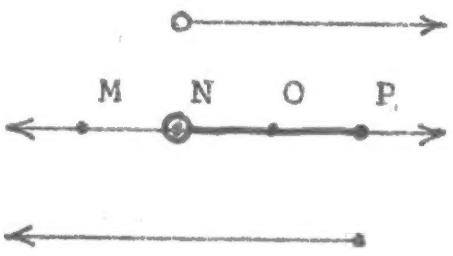


FIGURE 3

OBJECTIVE OF ACTIVITY III

pages included in the first booklet received by the students in T_3:

> You now can know in ADVANCE what is considered important by the author. You know in advance:
>
> 1. That the objective of this unit of instruction is that you will be able to perform tasks of the type described in step 5-A.
>
> 2. That the author expects you to learn the sub-tasks described in the Learning Sequence before you learn the final task (step 5-A).
>
> 3. That the author expects you to learn the sub-tasks and then the final task in the sequence shown on the opposite page [see Appendix C, page 304].

A complete copy of the materials given to the students in treatment T_3 is provided in Appendix C. The text material insert pages are placed at the points where the text material was provided in the activity booklets.

Besides the text material, the students in treatment T_4 received at the beginning of the unit of instruction a copy of the learning hierarchy, a set of examples of each task in the hierarchy, and descriptive comments as in treatment T_3. In addition, the T_4 students received at the beginning of each activity a written statement (with examples) of the behavioral objective of the activity and a written description of the activity's place in the learning hierarchy.

The behavioral objective for each activity was presented in T_4 in the same format as that used in treatment T_2. The following example of the type of comments provided in T_4 about the place of the activity in the learning hierarchy is from activity IV:

> The place of this lesson in the Learning Sequence is shown in red on the OPPOSITE PAGE. Lesson IV is designed to teach you the skills to perform the tasks described in steps 3-A and 3-B. Acquiring the ability to perform the tasks of 3-A and 3-B will enable you to learn to perform the task of step 4-A in Lesson V. You might wish to look now in the green folder at an example of step 4-A.
>
> You will be able to perform the tasks of steps 3-A and 4-A in this lesson because you have already acquired the skill to perform the tasks of steps 1-A and 1-B (shown in blue) [see Appendix D, page 368].

A copy of the materials given to the students in treatment T_4 is shown in Appendix D. As with the other treatments, the text material insert pages are placed at the points where the text material was provided in the activity booklets.

Criterion Instruments

Construction of the quizzes. Quizzes consisting of from two to four problems each were constructed for administration to each student in the experiment as he completed each of the activities. The quizzes were given for two reasons:

1. To provide data for a second validation of the learning hierarchy.
2. To provide data on the students' awareness of the objective of each activity in the instructional unit.

Since the first validation of the learning hierarchy was done under quite different teaching conditions than the teaching conditions of the experimental treatments, a decision was made to determine whether the hierarchy was still valid under the new teaching conditions. The findings from the new validation calculations are provided in Chapter IV.

The immediate purpose for informing students in this study of the behavioral objectives with and without the learning hierarchy was to enable them to be aware of the objectives of the activities in which they were engaged. A post-hoc analysis of the relationship between the students' level of awareness of the objectives and the students' performance on immediate and delayed posttests was included in the plans for this study. The design of this post-hoc analysis is described later in this chapter under Statistical Analysis. The quizzes were designed to not only provide a measure of the students' awareness of the objectives of the activities but also to motivate the students to pay attention to the information provided about the objectives in each activity booklet (see Appendix E, page 397). Those students in treatments T_2, T_3, and T_4 received a version of the quizzes which included instructions to identify and work only those quiz problems which were examples of the activities' objectives. Since one or more problems were included on each quiz which were not examples of the objective of the activity, students who knew the objectives of the activity did not have to work all of the problems which were given. A copy of the quiz given after activity V to each student in treatments T_2, T_3, and T_4 is shown as Figure 4 on the following page for illustrative purposes. A correct response on this quiz would have only included the circling of the number 3 and working problem 3. The students in treatment T_1 received quizzes containing the same problems with the written instructions deleted. The T_1 students worked all the problems on each quiz. Copies of all the quizzes of both versions are included in Appendix E.

The face validity of the quizzes was assured by having a one-to-one correspondence between the wording of the problems on the quizzes and the wording of the behavioral objectives in the learning hierarchy. The following comparison between the stated objective of sub-task 2-A in the learning hierarchy shown in Figure 1 on page 37 and

Lesson V--"Check-up" Questions

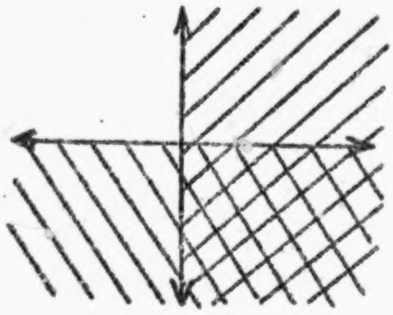
Name ______________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

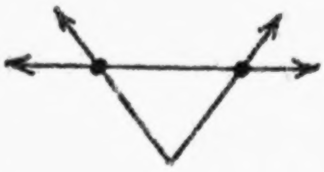
INSTRUCTIONS:

A. Circle the number of the problems which are examples of the objectives of this lesson.
B. Work only the problems you circled.

1. IDENTIFY, by coloring in red, the $\cap$ of the two following graphs:



2. NAME the $\cap$ set, shown in red, of the two following graphs. The $\cap$ set is to be named in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles:



3. NAME the solution set of the system of two graphs. The solution set is to be named in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

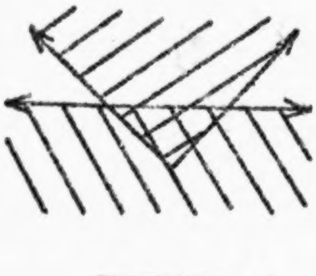


FIGURE 4

ACTIVITY V QUIZ

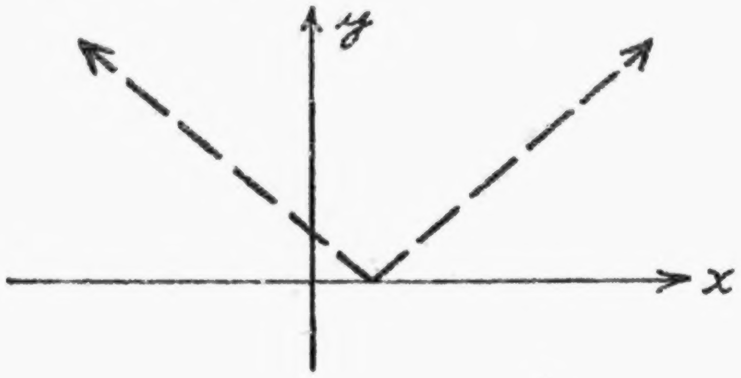
problem No. 3 on the quiz for activity VII-a illustrates this one-to-one correspondence. Activity VII-a was designed to teach the students the skill to perform the behavior described in sub-task 2-A. Problem No. 3 on the quiz for activity VII-a was designed to test the students' ability to perform the skill of sub-task 2-A after having been taught by the self-instructional materials in activity VII-a.

Sub-task 2-A:

> GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

Problem No. 3 on the quiz for activity VII-a:

> GIVEN the relation $\{(x,y) \mid y > |x-1|, x \text{ real}\}$ and its graph with the shaded region deleted; IDENTIFY by shading the region which satisfies $y > |x-1|$.



The problems on the quizzes which were not examples of the objectives of the activities were examples of problems which were used in the text material to establish the foundation for teaching the objective skills. Hence, it was assumed that the students who had access to the objective of each activity would be motivated by the nature of the quizzes to take advantage of the information on the objectives by differentiating as they studied the text material between the material that dealt directly with the objective and that material which played only a supportive role. The data obtained on the students' awareness of the objectives are displayed in Chapter IV.

Construction of the performance test. A performance test was constructed to determine whether or not the students in the experiment had acquired the terminal behavior of the learning hierarchy. The performance test was constructed for use as both the immediate posttest (achievement test) and the delayed posttest (retention test). The test consisted of seven problems, each of which tested the students' ability to perform the terminal task of the learning hierarchy. Three versions of the test were constructed to provide some assurance that the scores on the tests reflected the students own work. The three versions differed only in the order that the problems were listed. Copies of each version are included in Appendix F. The students' reactions to the material used in the experiment, along with administrative information, were requested on the cover page of each performance test. Their written reactions were not considered part of the performance test.

The face validity of the performance test was assured by a one-to-one correspondence between the wording of the test problems and the wording of the terminal objective in the learning hierarchy. One set of instructions was given for all seven problems. The following comparison between the terminal objective and the test instructions illustrates the one-to-one correspondence in wording:

Terminal task (5-A):

> GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Performance test instructions:

> GIVEN the system of two relations in each of the following problems, CONSTRUCT a graph of the solution set of each system and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Care was taken to include on the test every type of relation in the Set A referred to in the terminal task. The test included a minimum of two relations of each type. Only one type of relation in Set A was used more than three times on the test. The one exception was the type of relation involving absolute values, which was used four times.

The Kuder-Richardson procedure for estimating reliability was applied to the performance test scores to determine the reliability of the test under the four different treatments. The findings on test reliability are reported in Chapter IV.

SUBJECTS

The target population chosen for this research was the set of all elementary education major students who enroll in the second semester of a two-semester sequence of college mathematics courses designed to teach the content material required for elementary school teachers. The available population was the group of 103 students enrolled in four of the General Mathematics 205 sections at Towson State College in Towson, Maryland. The Math 205 course is designed for elementary education majors and is described by the college bulletin as including

> . . . elements of algebra, basic geometry, graphs, applications of percent, proportion and variation, right triangle relationships, logarithms, elementary statistics, and new topics in mathematics [2].

Fundamental Concepts of Arithmetic 204 is the prerequisite for Math 205. The Math 204 course includes:

> . . . origins of numbers, structure of a positional number system, principles underlying the fundamental operations, and computations with approximate numbers [3].

For most of the students in the experiment, the Math 205 course was their second college mathematics course.

The sample for this experiment, while chosen from the available population of 103 students, was essentially identical to this population. Those students within the population who had received a grade of A, B, C, or D for Math 204 were identified as the sample. Some students within the four sections were transfers from other colleges and, hence, grades for Math 204 were not available. A total of 92 students of the 103 in the population had grades from A to D in Math 204.

The students' grades in Math 204 provided a variable for blocking which was highly correlated with the responding variable (4). Hence, in order to increase the precision of the experiment (i.e., reduce the unaccountable variance), the students were blocked on levels of ability as reflected by the grades they made in Math 204. The ability levels were designated High for a grade of A in Math 204, Medium for a B in Math 204, and Low for either a C or D in Math 204. The blocking procedure resulted in 19 students in the High group, 36 in the Medium group, and 37 in the Low group.

Three students were randomly selected for deletion from the High group and one student was randomly selected for deletion from the Low group. The resulting sample for the experiment consisted of 88 students with a distribution of 16 students in the High group, 36 in the Medium group, and 36 in the Low group. The students in each of the three groups were then randomly assigned to each of the four treatments. There were 22 students in each of the four treatments. Table I shows the distribution of students to the various experimental conditions of the study.

TABLE I

CELL SIZES FOR T x L ANOVA DESIGN

		Treatments				
		T_1	T_2	T_3	T_4	
Ability Levels	High	$n_{11} = 4$	$n_{12} = 4$	$n_{13} = 4$	$n_{14} = 4$	$n_{1.} = 16$
	Medium	$n_{21} = 9$	$n_{22} = 9$	$n_{23} = 9$	$n_{24} = 9$	$n_{2.} = 36$
	Low	$n_{31} = 9$	$n_{32} = 9$	$n_{33} = 9$	$n_{34} = 9$	$n_{3.} = 36$
		$n_{.1} = 22$	$n_{.2} = 22$	$n_{.3} = 22$	$n_{.4} = 22$	

EXPERIMENTAL PROCEDURES

The unit of instruction, consisting of eight activities with two activities having two parts, was administered by the investigator for eight consecutive class days. The random assignment of students to treatments resulted in the four treatments being represented in each classroom. The students not included in the sample of the experiment received the materials of treatment T_1. On the ninth day of the experiment, the students were administered a previously announced performance test as an achievement test. Two weeks later the students received unannounced the same performance test as a retention test. Following the recommendation of Winer (5) each student received a different version of the performance test than the one he had received two weeks earlier. This was done to control the carry-over affects from the achievement test to the retention test. For clarity, henceforth in this paper the achievement test will also be referred to as the immediate posttest and the retention test will often be referred to as the delayed posttest.

On the first class day, the investigator taught all the students in each of the four classes how to read a learning hierarchy. This was done as a self-contained topic presented as an educational novelty. During the process of teaching the students how to read a learning hierarchy, no reference was made to the topic of the three-week instructional unit.

As an introduction to the three-week unit, the students were told that the self-instructional material that they would be using during the next three weeks had been developed at the University of Maryland. In addition, they were told that different versions of the material were being submitted for their use and appraisal. They were not told how the versions differed. The achievement test and the quizzes were announced at this time with the assurance that each would determine part of their final grade in the course. No mention was made of the scheduled retention test.

The full time instructors of each class were in attendance at the first session to introduce the investigator as the temporary instructor for the next three weeks. After the introductory remarks, the full time instructors left the classroom and did not attend any of the future sessions of the experiment.

Each activity of the self-instructional unit was given to the students in a booklet format. Each booklet was administered at the beginning of the class period. As the student completed the activity he returned it to the investigator. At no point was any student allowed to take a booklet home for study. After completing each activity at their own pace, each student was administered the quiz for the activity. However, quizzes were not given after activities I, II, and III. Activity I was a review of sets and their union and intersection. The learning hierarchy constructed for the study was based upon the assumption that the students in the study had the skill to identify the union and intersection of sets. Hence, the review in activity I was designed to provide support for this assumption and was not intended to help the students learn any skill in the hierarchy. Hence, a quiz was not given after activity I. Quizzes were not given after activities II and III because of a delay in preparation of the quizzes. The students' ability to perform behavioral objectives of activities II and III were determined by including in the quiz for activity VIII assessment tasks for activities II and III.

The dates on which each activity, immediate posttest (achievement test), and delayed posttest (retention test) were administered are listed below:

April 9	Introduction, Activity I
April 11	Activity II
April 14	Activity III
April 16	Activity IV
April 18	Activity V and VI
April 21	Activity VIIa
April 23	Activity VIIb
April 25	Activity VIIIa and VIIIb
April 28	Immediate Posttest
May 12	Delayed Posttest

The four sections of Math 205 used in the study met on Mondays, Wednesdays, and Fridays at the following times:

Section 1	10:00--10:50 A.M.
Section 2	11:00--11:50 A.M.
Section 3	12:00--12:50 P.M.
Section 4	1:00-- 1:50 P.M.

Sections 1, 2, and 4 met in the same building but in different rooms. Section 3 met in another building which was an eight-minute walk from the first building. Those students who required additional class time to complete their activity booklets were permitted to attend additional class sessions.

The topic selected for the study and the Spring semester schedule of Towson State College made it necessary to limit the period between the immediate posttest and the unannounced delayed posttest to two weeks instead of the proposed three weeks. During this two-week period the full time teachers did not review the mathematics topic used in the experimental study nor did they teach new material requiring the use of the skills learned during the experiment.

STATISTICAL ANALYSIS

As noted in Chapter II, this study was designed to investigate several pair-wise comparisons among treatment group means. The nine hypotheses stated in Chapter II on pages 24 and 25 can be grouped into two areas of concern. The first six hypotheses are concerned with pair-wise comparison of the effects of four instructional treatments on achievement. The last three hypotheses (H_7, H_8, and H_9) are concerned with pair-wise comparison of the effects of the four treatments on retention in terms of over-all performance and rate of forgetting. The treatment-by-level analysis of variance was employed to determine whether the data of this study supported the first six hypotheses. An evaluation of the data in terms of the last three hypotheses was made utilizing a repeated measures analysis design. This section of Chapter III deals with the use of these two statistical designs in the study.

Treatment-by-level Analysis of Variance

Hays (6) stated that

> For any particular comparison, the test is more powerful when the comparison is planned than when it is post-hoc. . . .

Because planned comparisons are more likely to detect differences when they exist than the various post-hoc methods, the hypotheses listed in Chapter II were stated before the data were collected for this study. Planned comparisons may be used in lieu of the over-all F test, while post-hoc comparisons are used as a supplement to an over-all F test which has shown significance. Hence, the five planned pairwise comparisons of the effects of the four treatments on achievement stated in hypotheses H_1 through H_5 were conducted instead of the over-all F test. In order to assure that the experimental design and the analysis were congruent, the schema recommended by Hopkins and Chadbourn (7) and Hays (8) for making c multiple comparisons among k treatment means was followed in this study.

In order to increase the precision of the experiment involving the planned contrasts (i.e., reduce the error term by accounting for some of the variance), the treatment-by-levels analysis of variance design (T x L ANOVA) was utilized. As previously mentioned, the students in the experiment were blocked on the levels of ability reflected by the grades they made in Math 204. The ability levels were designated High, Medium, and Low. The students with an A in Math 204 were placed in the High group, B in the Medium group, and those students with a C or D in the Low group. The students in each ability group were then randomly assigned to one of the four treatment groups.

In order for the T x L ANOVA to provide a valid test of the treatment effects on achievement, efforts were made to assure that the following assumptions were met:

1. The students from each of the ability levels are randomly assigned to the four treatments.
2. The variances of the student scores within the levels of ability are homogeneous.
3. The data are independent, i.e., a given response is in no way affected by other responses.
4. The students are drawn from a normal population.
5. The n's of the cells are equal or proportional.

The randomization assumption was met in the manner discussed in the preceding paragraph.

Homogeneity of variances was tested by Hartley's F_{max} test (9). The findings are included in Chapter IV.

Peckham and Hopkins have stated that independent responses:

> . . . are most adequately obtained by randomly assigning subjects to treatments, by administering identical treatments to individuals who are insulated from their peers during treatment, . . . [10].

The self-instructional format of the instructional materials provided for such insulation. It was assumed that the separate individual activity booklets would sufficiently insulate the students from the responses of their peers. Because the role of the investigator in the class sessions was only administrative, his presence did not alter the conditions established for independent responses.

Ferguson had this to say concerning the robustness of the assumption of normality:

> For large samples the normality of the distributions may be tested using a test of goodness of fit, although in practice this is rarely done. When the samples are fairly small, it is usually not possible to rigorously demonstrate lack of normality in the data. Unless there is reason to suspect a fairly extreme departure from normality, it is probable that the conclusions drawn from the data using an F test will not be seriously affected [11].

There appears, also, to be a specific advantage in the use of the analysis of variance with respect to the robustness of the normality assumption. Ferguson continued:

> One advantage of the analysis of variance is that reasonable departures from the assumptions of normality and homogeneity may occur without seriously affecting the validity of the inferences drawn from the data [12].

All of the students in the target population were elementary education majors enrolled in the second semester course of a two-course sequence in mathematics. Most of the students in the available population were freshmen and sophomores and all had met Towson State College's prerequisite requirement for taking Math 205. The assumption was made that the population from which the students were obtained was normal with respect to ability.

Table I on page 54 illustrates the planned proportional n's in the cells. One student dropped out of the study because of illness. She was in treatment T_1 and in the Low ability group. Ferguson stated in reference to cell frequencies for a two-way analysis of variance:

> Both the methods of expected equal and expected proportionate frequencies are in some degree approximate. Departure from equal n's in the former method and from proportionality of n's in the latter method will introduce some bias in the F test, the extent of the bias being related to the magnitude of the departures [13].

The loss of one student to this study was not considered a detriment to the proportional cell distribution. Since the magnitude of the departure from proportionality was so small, the possible resulting bias was considered to be negligible.

The decision was made before the analysis of the data to set the maximum probability of making a type-I error (α) at .05 .

An experiment-wise error rate was chosen by the investigator because it was considered desirable to be able to state that in all of the comparisons in this experimental study there was only an $\alpha = .05$ probability that a type-I error was made. Choosing an error rate that was experiment-wise meant that regardless of the number of permissable comparisons carried out, the probability was no more than .05 that one or more of the comparisons will turn out to be spuriously significant.

Repeated Measures Analysis

The repeated measures analysis was used to analyze the data obtained in this study to determine whether the data supported these three hypotheses:

> H_7--The four treatments have differential effects on the over-all performance of students.
>
> H_8--The four treatments result in different rates of forgetting.
>
> H_9--The differences in over-all performance scores resulting from the four treatments are not identically reflected at each ability level.

The decision to use the repeated measures analysis was supported by research conducted by Kenneth H. Wodtke of The Pennsylvania State University as part of a larger project funded by the U.S. Office of Education, under the Vocational Educational Act of 1963. In a journal article devoted to a consideration of several procedures for studying long-term retention, Wodtke (14) reported that the repeated measure design, described by Grant (15) and Winer (16), seemed most appropriate for the study of differential rates of forgetting and over-all performance. Wodtke stated that a statistically significant over-all effect between treatments (as reflected in the differences in the over-all performance scores)

> . . . would indicate that one instructional treatment was generally superior to another. A statistically significant treatment-by-retention measures

> interaction would indicate that the slopes of the retention curves in the treatment groups differed [17].

Similarly, a statistically significant levels-by-treatment interaction would indicate that the effects of the treatments on over-all performance were not identically reflected at each ability level.

Congruence between the experimental design of this study and the planned statistical analysis of the data with the repeated measures analysis was sought by the decision to follow the schema suggested by Hopkins and Chadbourn (18) and Hays (19) for making c multiple comparisons among k treatments. Based upon experimental conditions of this study and the Hopkins and Chadbourn schema, the Tukey-b method described by Winer (20) was included in the statistical design as the technique for locating significant differences between over-all performance means. In addition, the statistical design included the plotting of the slopes of the retention curves as a technique for determining which treatments resulted in the least rate of forgetting (i.e., improved retention). Inspection of the levels-by-treatment interaction profiles was included in the statistical design as the technique for obtaining an indication of the relative effects within ability levels resulting from the four treatments (21). These profiles consist of graphically depicted data showing changes in mean over-all performance scores across treatment methods separated by ability levels.

The three-factor repeated measures analysis design described by Winer (22) was used as the model for this study. The three factors involved were:

(a) the manipulated variable--the four instructional treatments;
(b) the normative variable--the levels of ability;
and (c) the responding variable--the posttests scores.

A schematic representation of the design is shown in Table II.

TABLE II

SCHEMATIC OF REPEATED MEASURES ANALYSIS DESIGN

		Posttests	
		Immediate	Delayed
	H_1	G_{H1}	G_{H1}
T_1	M_1	G_{M1}	G_{M1}
	L_1	G_{L1}	G_{L1}
	. . .		
	H_4	G_{H4}	G_{H4}
T_4	M_4	G_{M4}	G_{M4}
	L_4	G_{L4}	G_{L4}

Each of the groups was observed under both posttests, but each group was assigned to only one combination of treatment and level of ability. The notation G_{ij} denotes the group of subjects assigned to the i,j treatment combination where i denotes the ability level and j denotes the treatment number. A student within group G_{ij} would be identified by m(ij). This notation indicates that the subject effect was nested under both the treatment factor and the levels of ability factor. Since the subject factor (i.e., the variance of the characteristics peculiar to each student in the study) was nested under both the treatment factor and the levels of ability factors, there was no interaction between these latter factors and the subject factor.

The assumptions made for the repeated measures analysis were:

1. The students from each of the ability levels were randomly assigned to the four treatments.
2. The variances of the ability level means within the various treatments are homogeneous.
3. The variance of the "interaction of posttests and ability levels within treatments" is a pooling of the variance of the scores in each treatment group about the treatment mean after the effects of posttests and the effects of ability levels within treatments have been

subtracted. These variances for each treatment group are homogeneous.

4. The scores collected as data are normally distributed.

The assumptions of homogeneity of variance were tested by the F_{max} test. The findings are included in Chapter IV.

Planned Post-hoc Analysis

In addition to the planned post-hoc analyses outlined by the schema of Hopkins and Chadbourn and the plotting of the interaction profiles discussed in the previous section, another post-hoc analysis was made. Since the purpose of informing students in this study of the behavioral objectives with and without the learning hierarchy was to enable them to be aware of the objectives of the activities in which they were engaged, the following question is applicable: How did the students' level of awareness of the objectives affect their achievement test scores and over-all performance scores?

Data were obtained from the quizzes given after each activity in order to provide an answer to the above question. The students in treatments T_2, T_3, and T_4 were graded on the number of the problems on the quizzes that they correctly identified as examples of the objectives of the activities. The students were graded in the following manner: those who made a score of less than 25 were classified O_1, those with a score of 26-50 were classified O_2, those with 51-75 as O_3, and those with 76-100 as O_4. Ten points were added to their score for each problem correctly identified and ten points were subtracted from their score for each problem incorrectly identified. The score of each student determined in the above manner was referred to as the objectives test score for the student while that aspect of the quizzes which tested the students' awareness of the objectives was referred to as the objectives test.

An analysis of data obtained from the objectives test was made to determine the correlation between the students' awareness of the objectives and their achievement test scores. Also, a correlation coefficient which measured the degree of linear relation between the students' awareness of the objectives and their over-all performance scores was calculated. Both correlation coefficients are reported in Chapter IV.

The author expected the three treatments T_2, T_3, and T_4 to have differential effects on the students' awareness of the objectives. The analysis of variance statistical design was used to determine whether the data obtained from the objectives test supported this expectation. The findings from this analysis are presented in Chapter IV. Although there is no randomization involved in this type of analysis and, hence, no rigorous statement concerning statistically significant differences can be made, the comparisons provided some indication of the effect awareness of the objectives had on achievement test scores.

RESEARCH ASSUMPTIONS

In addition to the assumptions made with reference to the two statistical designs, the following assumptions were made for this study:

1. The learning hierarchy constructed as the basis for the instructional unit was a valid hierarchy at the 0.80 level for the consistency, adequacy, and completeness ratios.

2. If the students in the available population were given sets consisting of integers, they could identify (in written form) the intersection and/or union of any two sets. Also, if the student were given three Venn diagrams, they could identify (by shading) the intersection of the three diagrams.

3. The students could not perform the subordinate tasks on the bottom row of the learning hierarchy constructed for this study before they were subjected to the treatments.

4. The criterion instruments were valid measures of learned behaviors for the four treatments.

5. Any differences observed in the performance of the students in the four treatments were not a function of

the measuring instruments, but were dependent on the instructional methods employed.

6. The attitude and behavior of the principal investigator in his dealings with the students in the four treatment groups was unbiased.

7. The treatment groups are representative of the target population.

8. The treatments were applied as described.

9. The different experiences the students may have had in their respective classes prior to the experiment would not override the effect of the treatments. That is, classroom effect would not be a confounding factor.

10. Because additional class time was available to the slow student, the length of the self-instructional activities would not be a confounding factor.

CHAPTER III--FOOTNOTES

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[3] Ibid.

[4] Paul A. De Sena and Louise Ann Weber, "The Predictive Validity of the School College Ability Test (SCAT) and the American College Test (ACT) at a Liberal Arts College for Women," Educational and Psychological Measurement, XXV (Winter, 1965), 1149-51.

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[10] Percy D. Pecham and Kenneth D. Hopkins, "AERA Presession of Experimental Design and Analysis: The Experimental Unit in Statistical Analysis" (Laboratory of Education Research, University of Colorado), p. 1. (Mimeographed.)

[11] George A. Ferguson, Statistical Analysis in Psychology and Education (second edition; New York: McGraw-Hill Book Company, 1966), p. 294.

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[13] Ibid., p. 322.

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[15] D. A. Grant, "Analysis of Variance Tests in the Analysis and Comparison of Curves," Psychological Bulletin, LIII (1956), 141-54.

[16] Winer, Statistical Principles, pp. 337-47.

[17] Wodtke, "On the Assessment of Retention . . . ," p. 30.

[18] Hopkins and Chadbourn, "A Schema for Proper Utilization . . . ," p. 409.

[19] Hays, Statistics for Psychologists, p. 484.

[20] Winer, Statistical Principles, p. 87.

[21] Ibid., p. 346.

[22] Ibid., pp. 337-47.

CHAPTER IV

FINDINGS

The findings reported in this chapter in four sections are related to the following topics:

1. the validity of the learning hierarchy in the experimental setting,
2. the reliability of the posttest and the reliability of the objectives test,
3. the data analysis relevant to the nine research hypotheses stated in Chapter II, and
4. the post-hoc analyses of the objectives test data.

The third section is divided into two parts. The findings related to the comparison of the effects of the four treatments upon achievement scores are reported in the first part, entitled Treatment-by-Level Analysis of Variance. In the second part of the third section, entitled Repeated Measures Analysis, the findings pertaining to the comparison of the effects of the four treatments upon over-all performance and rate of forgetting are reported.

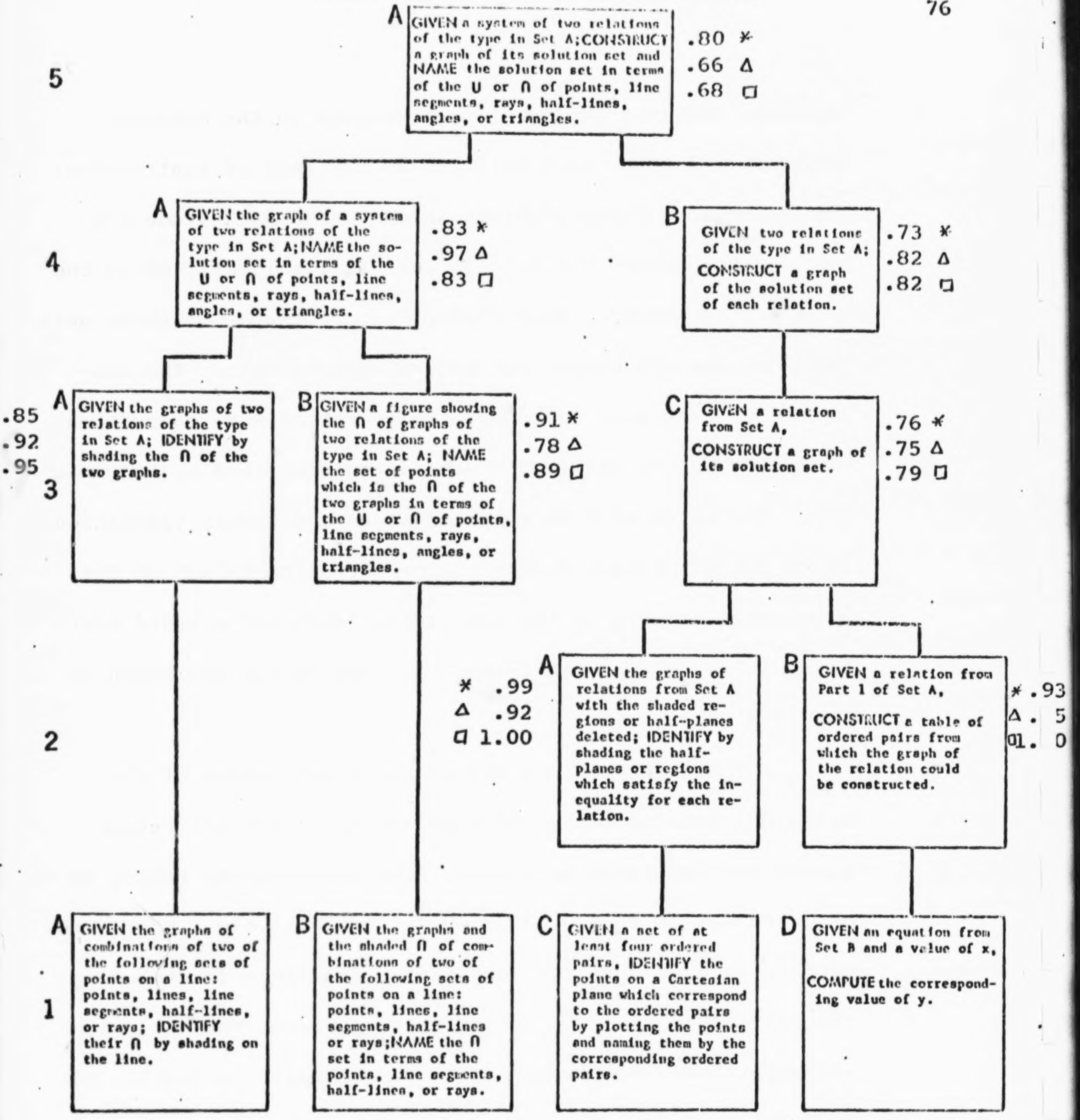
LEARNING HIERARCHY VALIDITY

The data for assessing the validity of the learning hierarchy in the instructional setting of the experiment were obtained from the students' performances on the quizzes administered after each activity in the unit of instruction. The quizzes included problems which tested each student's ability to perform the intellectual skills identified in the learning hierarchy. Each student's performance on these quiz problems was the source for the validation data. The consistency, adequacy, and completeness ratios were calculated according to the validation procedures described by Walbesser (1). The value of 0.80 selected as an acceptable validation level for all ratios during the construction trials of the hierarchy was used as the acceptable level for a valid hierarchy in the experimental setting. The ratios are shown in Figure 5 on page 76.

The data obtained during the construction of the hierarchy yielded ratios of 0.80 and above for all ratios except two completeness ratios. The completeness ratios at cells 5A and 4B were 0.67 and 0.71, respectively. The investigator assumed that the longer time available for teaching the unit of instruction during the experiment would result in completeness ratios above the 0.80 level at cells 5A and 4B. The experiment was thus conducted with the assumption that the hierarchy that was used in the experiment was valid at the 0.80 level. The data shown in Figure 5 do not support this assumption.

LEARNING SEQUENCE

(FINAL VALIDATION ANALYSIS RATIOS)


5
A
GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.
.80 *
.66 Δ
.68 □
4
A
GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.
.83 *
.97 Δ
.83 □
B
GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.
.73 *
.82 Δ
.82 □
3
.85
.92
.95
A
GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.
B
GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.
.91 *
.78 Δ
.89 □
C
GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.
.76 *
.75 Δ
.79 □
2
* .99
Δ .92
□ 1.00
A
GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.
B
GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.
* .93
Δ . 5
□1. 0
1
A
GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.
B
GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.
C
GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.
D
GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert, x \text{ real}\}$ $\{(x,y) \mid y \geq x + b, x \text{ real}\}$ $\{(x,y) \mid y \geq c - x, x \text{ reals}\}$	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$
Part 2. $\{(x,y) \mid y \geq d, x \text{ real}\}$ $\{(x,y) \mid x \geq e, x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

Legend of Ratios:

Consistency *
Adequacy Δ
Completeness □

FIGURE 5

Other findings which relate to the learning hierarchy are shown in Table III. The percent of students in the experiment who correctly responded to specific percentages of the problems on the immediate posttest are displayed. The percentages shown in the first row of Table III reflect that the posttest contained seven problems. Only 56% of the students in the experiment correctly worked 71% of the problems on the immediate posttest. The posttest problems were designed to measure the ability of the students to perform the terminal task of the learning hierarchy. The data indicate that the combination of the teaching techniques used in the experiment and the learning hierarchy did not adequately enable students to perform the final task.

TABLE III

PERCENT OF STUDENTS WORKING PROBLEMS CORRECTLY ON IMMEDIATE POSTTEST

Percent of Problems Correctly Worked	0%	14%	29%	43%	57%	71%	86%	100%
Percent of Students in Sample	100%	93%	85%	71%	62%	56%	44%	21%

CRITERION INSTRUMENT RELIABILITY AND OBJECTIVES TEST RELIABILITY

The Kuder-Richardson procedure for estimating reliability, using test-item statistics, was applied to the performance test scores of the sample used in the study. The Kuder-Richardson formula 20 measures internal consistency or homogeneity of the test material (2). An estimate of the reliability of the test under the four different treatments is provided by the reliability coefficients reported in Table IV for each of the treatment groups.

TABLE IV

RELIABILITY COEFFICIENTS OF THE POSTTEST FOR EACH TREATMENT

	Treatment Groups			
Measure	T_1	T_2	T_3	T_4
Kuder-Richardson estimated reliability	.76	.84	.79	.81

In his discussion related to analysis and evaluation of science tests, Hedges provides an answer to the question concerning an acceptable value for the reliability of science tests. Hedges (3) states,

> Your own tests should have a reliability of about .60, whereas standardized test-makers should get as high as .90 or more.

He also adds,

> If you ever get as high as .67 on one of your tests, be grateful.

On the basis of the criteria suggested by Hedges, the reliability coefficients displayed in Table IV for each treatment provide sufficient evidence to support the assumption of the reliability of the posttest.

The Kuder-Richardson formula 20 was also used to estimate the reliability of the objectives test scores. A reliability coefficient of 0.56 was obtained. The author considered this value sufficiently close to 0.60 to provide evidence to support the assumption of the reliability of the objectives test.

COMPARISON OF EXPERIMENTAL TREATMENTS

Statistical procedures utilized in this study to compare the effects on learning and retention of the four treatments include the treatments-by-levels analysis of variance and the repeated measures analysis. The findings from these two analyses are reported in this section. The research hypotheses are restated preceding the null hypotheses tested. After the relevant findings are reported, a statement of the appropriate statistical decision concludes each presentation.

In cases of missing data in the comparison of the treatments, a decision was made to use an estimate of the missing scores. One student became ill during the experiment and dropped out of the study. Four other students were absent for the unannounced delayed posttest. Winer stated in regard to the estimating of missing data:

> In cases where the number of observations within a particular cell is small relative to the number of observations in other cells in the same row and column or adjacent rows and adjacent columns, the information provided by these other cells may be used in estimating the mean of a specified cell [4].

The mean of each cell containing the missing data was used as an estimate of the missing data in the author's study.

Treatments-by-levels Analysis of Variance

The comparative effectiveness of the four instructional methods on student achievement was studied by treatments-by-levels analysis of variance. The findings relative to achievement pertain to the first six research hypotheses.

> Research Hypothesis 1: Giving students statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2) results in higher achievement scores for the students so informed than those students who are given no information beyond the actual instructional unit activity (T_1).

This hypothesis was examined by testing the statistical null hypothesis:

$$H_o^1: \mu_2 \leq \mu_1$$

Where μ_1 = mean of the immediate posttest scores for the T_1 group,
μ_2 = mean of the immediate posttest scores for the T_2 group.

> Research Hypothesis 2: Giving students copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3) results in higher achievement scores for the students so informed than for those students who are not given any information beyond the actual instructional unit activity (T_1).

This hypothesis was examined by testing the statistical null hypothesis:

$$H_o^2: \mu_3 \leq \mu_1$$

Where μ_1 = mean of the immediate posttest scores for the T_1 group,
μ_3 = mean of the immediate posttest scores for the T_3 group.

> Research Hypothesis 3: Giving students statements of the behavioral objectives (with examples) and copies of the learning sequence before each activity of an instructional unit (T_4) results in higher achievement scores for the students so informed than for those who are not given any information beyond the actual instructional unit activity (T_1).

This hypothesis was examined by testing the statistical null hypothesis:

H_o^3: $\mu_4 \leq \mu_1$.

Where μ_1 = mean of the immediate posttest scores for the T_1 group,

μ_4 = mean of the immediate posttest scores for the T_4 group.

> Research Hypothesis 4: Giving students statements of the behavioral objectives (with examples) and copies of the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2).

This hypothesis was examined by testing the statistical null hypothesis:

H_o^4: $\mu_4 \leq \mu_2$

Where μ_2 = mean of the immediate posttest scores for the T_2 group,

μ_4 = mean of the immediate posttest scores for the T_4 group.

> Research Hypothesis 5: Giving students statements of the behavioral objectives (with examples) and the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3).

This hypothesis was examined by testing the statistical null hypothesis:

H_o^5: $\mu_4 \leq \mu_3$

Where μ_3 = mean of the immediate posttest scores for the T_3 group,
μ_4 = mean of the immediate posttest scores for the T_4 group.

Research Hypothesis 6: The differences in achievement scores resulting from the four treatments are not identically reflected at each ability level.

This hypothesis was examined by testing the statistical null hypothesis:

H_o^6: $\theta_{jk} = 0$, for all jk

Where θ_{jk} = the interaction of the j^{th} level of ability and the k^{th} treatment.

A treatments-by-levels analysis of variance (T x L ANOVA) was employed to test each of the above six null hypotheses. The procedure for testing the first five null hypotheses included two steps:

1) Using T x L ANOVA, determine whether the data supported or rejected the null hypothesis: $\mu_1 = \mu_2 = \mu_3 = \mu_4$;
2) If the null hypothesis ($\mu_1 = \mu_2 = \mu_3 = \mu_4$) was rejected, by using the Tukey-b method, determine which of the first five null hypotheses were rejected by the data.

A summary of the results of the treatments-by-levels analysis of variance of immediate posttest scores appears in Table V. The F ratio observed for treatment effect was 0.563.

TABLE V

TREATMENTS-BY-LEVELS ANALYSIS OF VARIANCE OF IMMEDIATE POSTTEST SCORES

Source of Variation	df	Mean Square	F	Critical Value of F at .05 Level
Levels of Ability	2	6,503.54	--	
Treatments	3	282.09	.56 ns	2.78 (3,76 df)
Interaction (Treatments x Levels)	6	93.28	.19 ns	2.27 (6,76 df)
Within Cells (Error)	76	501.10		

The critical value of F at 3,76 df and 0.05 level of significance is 2.78 . Therefore, the over-all null hypothesis

$$H_o: \mu_1 = \mu_2 = \mu_3 = \mu_4$$

was retained. Since the over-all null hypothesis was retained, obviously no differences would be detected between pairs of sample means. Hence, each of the first five null hypotheses

$$H_o^1: \mu_2 \leq \mu_1$$

$$H_o^2: \mu_3 \leq \mu_1$$

$$H_o^3: \mu_4 \leq \mu_1$$

$$H_o^4: \mu_4 \leq \mu_2$$

$$H_o^5: \mu_4 \leq \mu_3.$$

was retained. No data were analyzed using the Tukey-b method.

The F ratio observed for ability-by-treatment interaction effect was 0.19 . The critical value of F at 6,76 df and 0.05 level of significance is 2.27 . Therefore, the null hypothesis

$$H_o^6: \theta_{jk} = 0, \text{ for all } jk$$

was retained.

The standard deviations and the cell and marginal means of the four treatment groups on the immediate posttest are reported in Table VI. The F_{max} test was used to test the degree of homogeneity of the sample variances of the achievement scores. The F_{max} observed was 116.93 and the F_{max} critical was 8.85 at the 0.05 level of significance. This finding challenges the appropriateness of the T x L ANOVA statistical design for the author's study. Support for the appropriateness of the design is provided by the classical Norton (5) study of the effects of heterogeneity of variance. Norton found that the analysis of variance test was robust with regard to violations of the homogeneity of variance assumption. Ferguson (6), also, argued for the robustness of the ANOVA test to departures from the homogeneity assumption.

TABLE VI

MEANS, CELL SIZES, AND STANDARD DEVIATIONS
OF IMMEDIATE POSTTEST SCORES

Ability Levels		Treatment				
		T_1	T_2	T_3	T_4	
	High	$\bar{X}_{11} = 95.50$ $n_{11} = 4$ $SD_{11} = 3.00$	86.50 4 17.99	95.00 4 10.00	91.00 4 18.00	$\bar{X}_{1.} = 92.00$ $n_{1.} = 16$
	Medium	$\bar{X}_{21} = 82.00$ $n_{21} = 9$ $SD_{21} = 14.11$	78.44 9 11.74	74.44 9 23.60	84.56 9 16.85	$\bar{X}_{2.} = 79.86$ $n_{2.} = 36$
	Low	$\bar{X}_{31} = 62.56$ $n_{31} = 9$ $SD_{31} = 26.81$	53.11 9 31.51	62.00 9 32.44	64.22 9 23.14	$\bar{X}_{3.} = 60.47$ $n_{3.} = 36$
		$\bar{X}_{.1} = 76.50$ $n_{.1} = 22$	$\bar{X}_{.2} = 69.54$ $n_{.2} = 22$	$\bar{X}_{.3} = 73.09$ $n_{.3} = 22$	$\bar{X}_{.4} = 77.41$ $n_{.4} = 22$	$\bar{X}_T = 74.14$ $N = 88$

Repeated Measures Analysis

The effectiveness of the four instructional methods on student over-all performance and rate of forgetting was studied using a repeated measures analysis. The findings relative to over-all performance and rate of forgetting pertain to the last three research hypotheses.

> Research Hypothesis 7: The four treatments have differential effects on the over-all performance of students.

This hypothesis was examined by testing the statistical null hypothesis:

$$H_o^7: \alpha_k = 0, \text{ for all } k$$

Where α_k = the effect of being in the k^{th} treatment group

> Research Hypothesis 8: The four treatments result in different rates of forgetting.

This hypothesis was examined by testing the statistical null hypothesis:

$$H_o^8: \alpha\gamma_{ik} = 0, \text{ for all } ik$$

Where $\alpha\gamma_{ik}$ = the interaction of the i^{th} posttest and the k^{th} treatment

> Research Hypothesis 9: The differences in over-all performance scores resulting from the four treatments are not identically reflected at each ability level.

This research hypothesis was examined by testing the statistical null hypothesis:

H_o^9: $\beta\gamma_{jk} = 0$, for all jk

Where $\beta\gamma_{jk}$ = the interaction of the j^{th} level of ability and the k^{th} treatment.

A repeated measures analysis was used to test each of the above three hypotheses. The procedures for testing the eighth null hypothesis included the following two steps:

1) Using the repeated measures analysis, determine whether the data supported or rejected the null hypothesis: $\alpha\gamma_{ik} = 0$, for all ik .
2) If the null hypothesis ($\alpha\gamma_{ik} = 0$, for all ik) was rejected, determine which treatments resulted in the least rate of forgetting by plotting the slopes of the retention curves.

Similarly, the procedure for testing the ninth null hypothesis included these two steps:

1) Using the repeated measures analysis, determine whether the data supported or rejected the null hypothesis: $\beta\gamma_{jk} = 0$, for all jk .
2) If the null hypothesis ($\beta\gamma_{jk} = 0$, for all jk) was rejected, plot the interaction profiles consisting of graphically depicted data showing changes in over-all performance scores across treatment methods separated by ability levels.

A summary of the results of the repeated measures analysis of immediate and delayed posttests scores is shown in Table VII. The F ratio observed for treatment effect was 0.37 . The critical value of F at 3,76 df and 0.05 level of significance is 2.73 . Hence, the null hypothesis

TABLE VII

SUMMARY OF REPEATED MEASURES ANALYSIS

Source of Variation	df	SS	MS	F	F at .05
Between Subjects	87				
Levels	2	32,147.89	16,073.95	--	
Treatments	3	939.29	313.10	.37 ns	2.73
Interaction: (Level x Treatment)	6	1,426.14	2,139.21	2.50*	2.23
Subj w. groups	76	64,974.23	854.92		
Within Subjects	88				
Posttests	1	22.55	22.55	--	
Interaction: (Level x Posttests)	2	417.67	835.34	5.44*	3.13
Interaction: (Treatment x Posttests)	3	366.70	1,100.09	7.16*	2.72
Interaction: (Level x Treatment x Posttest)	6	332.20	1,993.22	12.98*	2.23
Posttests x subj w. groups	76	11,669.38	153.54		
Total	175	112,296.05			

*Significant at the .05 level of significance.

$$H_o^7: \alpha_k = 0, \text{ for all } k$$

was retained. The F ratio observed for treatment-by-posttest interaction effect was 2.50 . Since the critical value of F at 6,76 df and 0.05 level of significance is 2.23 , the null hypothesis

$$H_o^8: \alpha\gamma_{ik} = 0, \text{ for } ik$$

was rejected. The fact that there was a statistically significant treatment-by-posttests interaction indicated that the slopes of the retention curves for the treatment groups were different. In order to determine which treatments resulted in the least rate of forgetting, the slopes of the retention curves were plotted as shown in Figure 5 based upon the data in Table VIII. The retention curve of treatment T_4 had a slope which approached a value of zero, while the retention curve of treatment T_2 had a positive slope.

TABLE VIII

TREATMENT MEANS FOR POSTTESTS

	Immediate	Delayed
Treatment T_1	76.50	72.95
Treatment T_2	69.55	73.50
Treatment T_3	73.09	70.50
Treatment T_4	77.50	76.80

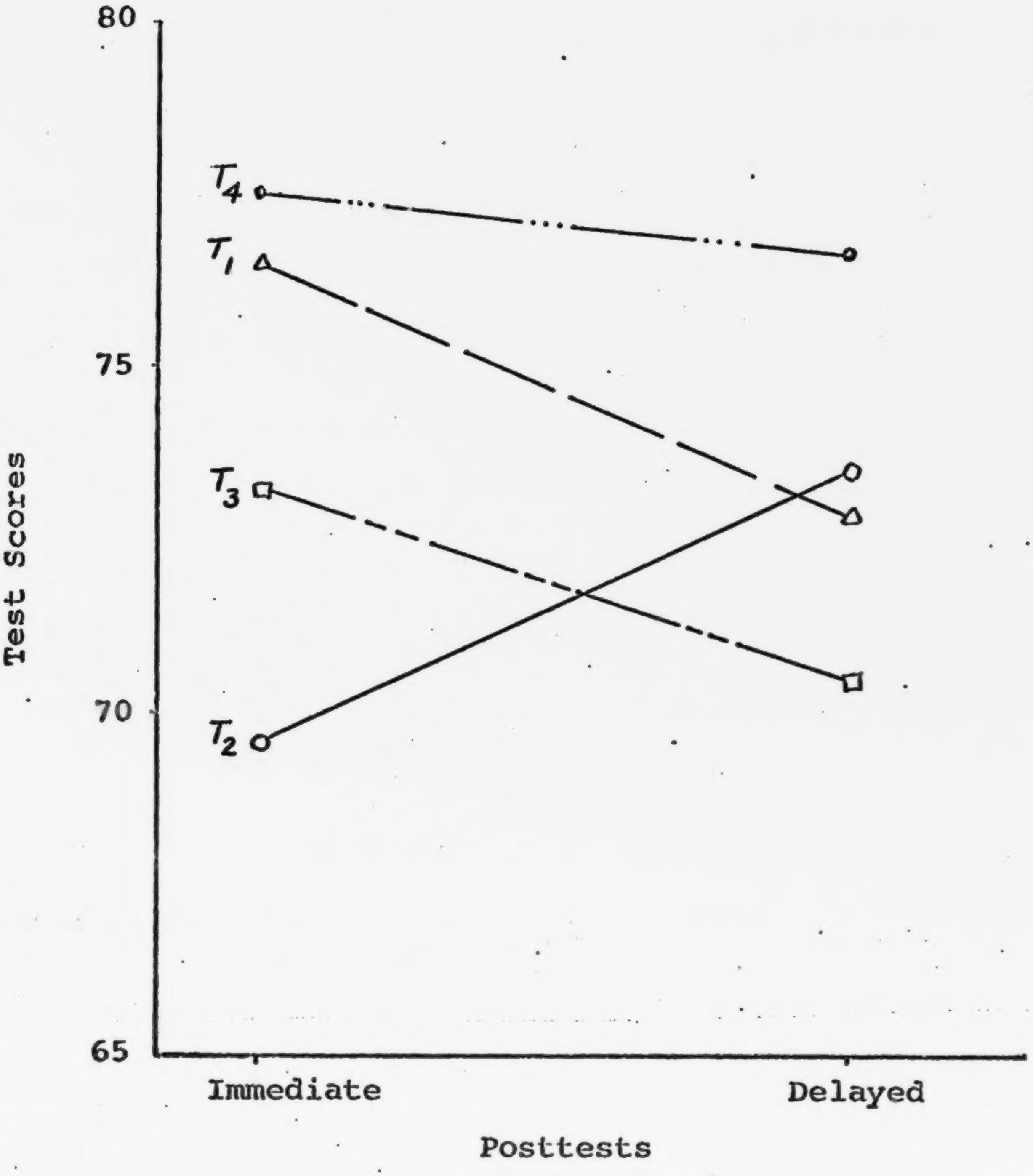


FIGURE 6

RETENTION CURVES

The F ratio observed for level-by-treatment effect was 2.50 . The critical value of F at 6,76 df and 0.05 level of significance is 2.23 . Therefore, the null hypothesis

$$H_o^9: \beta\gamma_{jk} = 0, \text{ for } jk$$

was rejected. The level-by-treatment interaction profiles are shown in Figure 6. The profiles are based upon the data in Table IX. The mean of the middle ability students in treatment T_4 was 87.14 , while the mean across all four treatments was 81.01 . The means of the other two ability groups in treatment T_4 did not show as much difference between their respective means and their means across the four treatments.

TABLE IX

ABILITY LEVEL OVER-ALL PERFORMANCE MEANS FOR TREATMENTS

Ability Levels	Treatments			
	T_1	T_2	T_3	T_4
High	94.25	86.75	94.50	94.00
Medium	82.11	81.44	73.33	87.17
Low	58.67	54.83	60.00	59.66

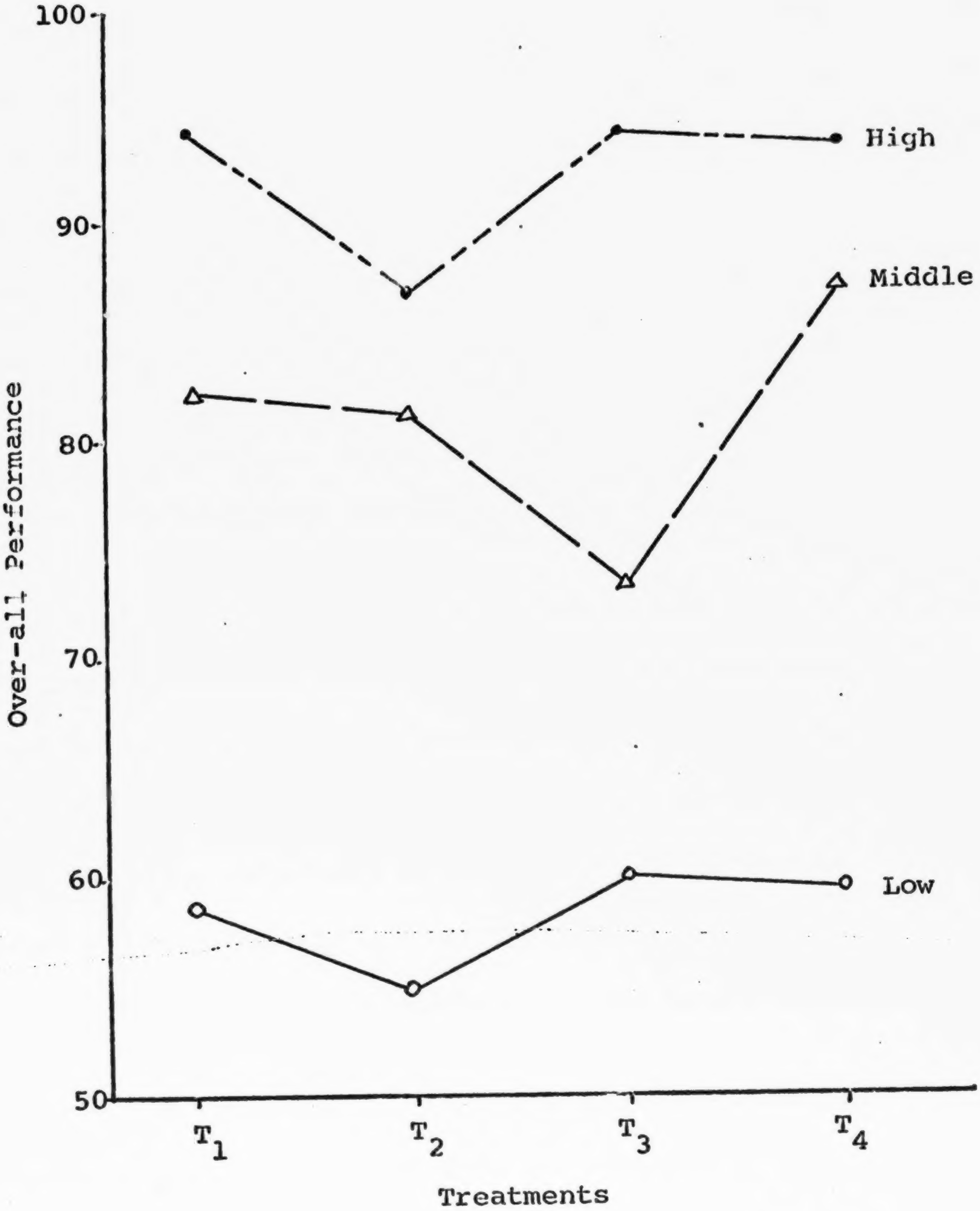


FIGURE 7

LEVEL x TREATMENT INTERACTION PROFILES

Table X presents the standard deviation and the cell and marginal means for the immediate posttest and the delayed posttest scores analyzed in the repeated measures analysis. An analysis of the display of standard deviations in Table VI reveals a noticeable heterogeneity of variances. The discussion on page 85 concerning the robustness of the analysis of variance in regard to the assumption of homogeneity of variances applies in this case, also.

POST-HOC ANALYSES

The findings obtained from both planned and unplanned post-hoc analyses are reported in this section. The planned post-hoc analysis consisted of an analysis of the objective test scores while the unplanned post-hoc analysis involved an analysis of the data to determine percentages of students in each treatment who correctly responded to the same number of problems on the performance test.

Planned Post-hoc Analysis

Students in this study were given statements of the behavioral objectives with and without copies of the learning hierarchy to enable them to be aware of the objectives of the activities in which they were engaged. An analysis of data obtained from the objective test portion of the quizzes

TABLE X

MEANS, CELL SIZES, AND STANDARD DEVIATIONS OF IMMEDIATE AND DELAYED POSTTEST SCORES

Treatments	Levels of Ability	Posttests		
		Immediate	Delayed	
T_1	H1	$\bar{X}_{111} = 95.50$ $SD_{111} = 3.00$ $n_{111} = 4$	$\bar{X}_{112} = 93.00$ $SD_{112} = 8.08$ $n_{112} = 4$	$\bar{X}_{11.} = 94.25$
	M1	$\bar{X}_{121} = 82.00$ $SD_{121} = 14.11$ $n_{121} = 9$	82.22 14.16 9	$\bar{X}_{12.} = 82.11$
	L1	62.56 26.81 9	54.77 24.94 9	$\bar{X}_{13.} = 58.67$
T_2	H2	86.50 17.99 4	87.00 9.87 4	$\bar{X}_{21.} = 86.75$
	M2	78.44 11.74 9	84.44 15.90 9	$\bar{X}_{22.} = 81.44$
	L2	53.11 31.52 9	56.56 21.28 9	$\bar{X}_{23.} = 54.83$

TABLE X (continued)

Treat-ments	Levels of Ability	Posttests		
		Immediate	Delayed	
T_3	H3	$\bar{X}_{131} = 95.00$ $SD_{131} = 10.00$ $n_{131} = 4$	$\bar{X}_{132} = 94.00$ $SD_{132} = 8.49$ $n_{132} = 4$	$\bar{X}_{31.} = 94.50$
	M3	74.44 23.60 9	72.56 25.06 9	$\bar{X}_{32.} = 73.33$
	L3	62.00 32.44 9	58.00 35.44 9	$\bar{X}_{33.} = 60.00$
T_4	H4	91.00 18.00 4	97.00 3.47 4	$\bar{X}_{41.} = 94.00$
	M4	84.56 16.85 9	89.56 18.78 9	$\bar{X}_{42.} = 87.17$
	L4	64.23 23.14 9	55.11 30.92 9	$\bar{X}_{43.} = 59.66$
		$\bar{X}_{..1} = 74.14$	$\bar{X}_{..2} = 73.44$	$\bar{X}_T = 73.80$

which were administered after completion of each activity) was made to determine the correlation between the students' awareness of the objectives and their achievement test scores. Students in treatments T_2, T_3, and T_4 were graded on the number of problems on the quizzes that they correctly identified as examples of the objectives of the activities. The students were then classified into four categories O_1, O_2, O_3, and O_4 according to their grades on the objectives test. The score means in the four classifications and the number of students in each classification are shown in Table XI. There was a total of 66 students in treatments T_2, T_3, and T_4. The objectives test scores of eight of the 66 students were not included in the data displayed in Table XI because the eight students were inadvertently administered one quiz designed for the students in treatment group T_1.

TABLE XI

OBJECTIVES TEST CLASSIFICATIONS--
MEANS, NUMBER OF STUDENTS

	Classifications			
	O_1 (0-25 pts)	O_2 (26-50 pts)	O_3 (51-75 pts)	O_4 (76-100 pts)
Means	10.00	40.00	63.75	92.00
No. of students	7	8	8	35

A comparison of the means of the objectives test scores and the corresponding achievement test scores and over-all performance scores of the students in each of the four objectives classifications is shown in Table XII. Two Pearson product-movement correlation coefficients were calculated. The correlation coefficient measuring the degree of linear association between the students' awareness of the objectives and their performance on the achievement test was found to be 0.11 . The correlation coefficient which measured the degree of linear relation between the students' awareness of the objectives and their over-all performance was found to be 0.40 . The findings reveal that the students' awareness of the objectives was uncorrelated with achievement test scores and also was uncorrelated with the students' over-all performance.

TABLE XII

COMPARISON OF MEANS--OBJECTIVES TEST, ACHIEVEMENT TEST, AND OVER-ALL PERFORMANCE

Classification	Objectives Test Means	Achievement Test Means	Over-all Performance Means
O_1	10.00	69.43	139.43
O_2	40.00	64.00	130.75
O_3	63.75	60.50	120.88
O_4	92.00	81.14	162.03

Different procedures were used to inform the students in the three treatments, T_2, T_3, and T_4, of the behavioral objectives. These procedures are described in Chapters II and III. From the nature of the three procedures, the author expected the three treatments to have differential effects on the students' awareness of the objective. The analysis of variance statistical design was utilized to test the following null hypothesis:

$$H_o: \mu_2^* = \mu_3^* = \mu_4^*$$

Where μ_2^* = the mean of students' objectives test scores in treatment T_2.

μ_3^* = the mean of students' objectives test scores in treatment T_3.

μ_4^* = the mean of students' objectives test scores in treatment T_4.

Table XIII presents the size, means, variances, and standard deviations of the three treatment groups on the objectives test. One student's score was randomly selected for deletion from treatment T_4 in order to obtain equal cell sizes. Whereas the mean of treatment T_3 was 48.5 , treatment T_4 scored an average of 65.4 points out of 100 points.

Table XIV presents the summary table for the analysis of variance. The F ratio observed for treatment effect was 10.66 while the F critical for 2,55 degrees of freedom was 5.79 at the 0.005 level of significance. Therefore, the

TABLE XIII

SIZE, MEANS, VARIANCES, AND STANDARD DEVIATIONS OF THE THREE TREATMENT GROUPS (T_2, T_3, T_4) ON THE OBJECTIVES TEST

Treatments	n	Mean	Variance	Standard Deviation
T_2	19	80.0	533.33	23.1
T_3	19	48.5	914.22	30.2
T_4	19	85.4	650.22	25.5

TABLE XIV

TREATMENTS-BY-LEVELS ANALYSIS OF OBJECTIVES TEST SCORES

Source of Variation	df	Mean Square	F	F_c at .005 Level
Treatments	2	7,302.5	10.06*	5.79
Error	55	726.0		
Total	57	8,028.5		

*p < .005

null hypothesis

$$H_o: \mu_2^* = \mu_3^* = \mu_4^*$$

was rejected. The Tukey-b method was applied to locate differences between means. The following findings were obtained at 0.05 level of significance:

$$\mu_2^* > \mu_3^*$$

$$\mu_2^* = \mu_4^*$$

$$\mu_4^* > \mu_3^*$$

Since there is no randomization involved in this quasi-experimental analysis, the author cannot make a rigorous statement concerning significant differences. However, the comparisons provided an indication of the effect the three treatments had on the students' awareness of the objectives.

To test for homogeneity of variances, the F_{max} test was used. The following null hypothesis was tested:

$$H_o: \sigma_2^2 = \sigma_3^2 = \sigma_4^2$$

The F_{max} observed was 1.71 and the F_{max} critical was 2.21 at the 0.05 level of significance. Thus, the null hypothesis was retained and the assumption of homogeneity of variances was supported by the data.

Other Post-hoc Analyses

The findings shown in Table XV show the percentage of students in each treatment who correctly worked specific numbers of problems on the achievement test. Only 10% of the students in treatment T_1 worked all seven of the problems correctly, while 32% of the students in treatment T_4 worked all seven of the problems correctly.

TABLE XV

PERCENTAGE OF STUDENTS WORKING PROBLEMS ON ACHIEVEMENT TEST

	Number of Problems Correctly Worked							
	0	1	2	3	4	5	6	7
T_1, cumulative percentage of students	100	95	90	76	66	57	47	10
T_2, cumulative percentage of students	100	86	82	59	50	45	36	14
T_3, cumulative percentage of students	100	91	86	73	64	64	41	27
T_4, cumulative percentage of students	100	100	86	77	68	59	50	32

SUMMARY OF FINDINGS

A comparison of the four treatment means of the immediate posttests scores detected no significant difference at the 0.05 level of significance. Also, an analysis of the immediate achievement scores did not yield a significant interaction effect between levels of ability and treatments.

No significant differences were found between the over-all performance means of the four treatments. However, analysis of the over-all performance data detected a significant levels-by-treatment interaction. In addition, a significant difference in rate of forgetting resulting from the four treatments was observed. While the slopes of treatments T_1 and T_3 retention curves had negative gains and the slope of the retention curve of treatment T_2 approached zero, the slope of the treatment T_1 retention curve was found to have a positive gain.

The means of the objectives test means of treatments T_2 and T_4 were each observed to be significantly greater than the objectives test mean resulting from treatment T_3. The percentages of students in treatments T_2 and T_4 who worked all seven of the problems on the achievement test correctly were 10 percent and 32 percent, respectively.

CHAPTER IV--FOOTNOTES

[1] Henry H. Walbesser, "A Hierarchically Based Test Battery for Assessing Scientific Inquiry" (Paper presented at American Educational Research Association Annual Meeting, 1968).

[2] George A. Ferguson, _Statistical Analysis in Psychology and Education_ (second edition; New York: McGraw-Hill Book Company, 1966), pp. 379-80.

[3] William D. Hedges, _Testing and Evaluation for the Sciences_ (Belmont, California: Wadsworth Publishing Company, Inc., 1966), 248 pp.

[4] B. J. Winer, _Statistical Principles in Experimental Design_ (New York: McGraw-Hill Book Company, 1962), p. 281.

[5] E. F. Lindquist, _Design and Analysis of Experiments in Psychology and Education_ (Boston: Houghton Mifflin Company, 1953), pp. 78-90.

[6] Ferguson, _Statistical Analysis_, p. 86.

CHAPTER V

SUMMARY, CONCLUSIONS, AND IMPLICATIONS

In this chapter the study is summarized, conclusions are outlined and discussed, and certain implications are suggested. In addition to the consideration of the conclusions, the validity of the study is considered in the discussion. The implications considered in the final portion of the chapter are derived from the findings of this study and pertain to both current practice and future research.

SUMMARY

This study was conducted to investigate the question:

> If a group of students is informed of the behavioral objectives and the learning hierarchy of a unit of instruction and another group of students receiving the same unit of instruction is not so informed, will there be differences in effect on learning and retention?

It was expected that those students who are informed of the behavioral objectives of an activity will perform higher on achievement and retention posttests than those students who are not so informed. Moreover, it was expected that students who are informed of the activity's place in a hierarchical learning sequence (designed after Robert Gagné's cumulative learning model) in addition to being informed of the behavioral objectives of the activity will perform higher on achievement and retention posttests than those who are informed just of the behavioral objectives of the activity. The author's research was designed to determine whether for a specific population with specific treatments data could be obtained to support the above expectations. Accordingly, eighty-eight elementary education majors in a four-year college were blocked on ability levels and randomly assigned to four treatments. While receiving different information about the behavioral objectives and the hierarchical learning sequence, all four groups received the same set of self-instructional text material covering a mathematical unit of instruction. The following research hypotheses, associated with the initial question and reflecting the stated expectations, were tested by the experiment:

> Research Hypothesis 1: Giving students statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2) results in higher achievement scores for the students so informed than those students who are given no information beyond the actual instructional unit activity (T_1).

> Research Hypothesis 2: Giving students copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3) results in higher achievement scores for the

students so informed than for those students who are not given any information beyond the actual instructional unit activity (T_1).

Research Hypothesis 3: Giving students statements of the behavioral objectives (with examples) and copies of the learning sequence before each activity of an instructional unit (T_4) results in higher achievement scores for the students so informed than for those who are not given any information beyond the actual instructional unit activity (T_1).

Research Hypothesis 4: Giving students statements of the behavioral objectives (with examples) and copies of the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only statements of the behavioral objectives (with examples) before each activity of an instructional unit (T_2).

Research Hypothesis 5: Giving students statements of the behavioral objectives (with examples) and the learning hierarchy before each activity of an instructional unit (T_4) results in higher achievement scores for students so informed than for students who are given only copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit (T_3).

Research Hypothesis 6: The differences in achievement scores resulting from the four treatments are not identically reflected at each ability level.

Research Hypothesis 7: The four treatments have differential effects on the over-all performance of students.

Research Hypothesis 8: The four treatments result in different rates of forgetting.

> Research Hypothesis 9: The differences in over-all performance scores resulting from the four treatments are not identically reflected at each ability level.

The subjects were students enrolled in four sections of the second semester of a two-semester sequence mathematics course for elementary education majors at Towson State College, Baltimore, Maryland in the spring of 1969. The students were classified according to their ability levels as reflected by their grades in the first semester of the two-semester sequence course before being randomly assigned to the following four treatments:

T_1--self-instructional text material on a mathematical unit.

T_2--self-instructional text material on a mathematical unit with the objectives given before each activity in the unit.

T_3--self-instructional text material on a mathematical unit with students informed of the learning hierarchy at the beginning and at the end of the unit.

T_4--self-instructional text material on a mathematical unit with students informed:
(a) at the beginning of the unit of the learning hierarchy for the instructional unit, and
(b) at the beginning of each activity of the objectives of that activity and of that activity's place in the hierarchical learning sequence.

The self-instructional material for the instructional unit, based upon the learning hierarchy constructed for this study and consisting of eight activities, was administered by the experimentor for eight consecutive class days. After the completion of the instructional unit, posttests were administered immediately to compare the degree of learning, and, after two weeks, to compare the amount of retention.

Through treatments-by-levels analysis of variance, the relative effectiveness of the four treatments on student achievement as measured by the immediate posttest was determined. Differential effects of the four treatments on learning the intellectual skill of the terminal task of the learning hierarchy were not significant at the 0.05 level. The treatments-by-levels interaction effects of the four treatments were also found not significant at the 0.05 level.

At a 0.05 level of significance, a repeated measures analysis detected no differential effects, resulting from the four treatments, on student over-all performance. However, the four treatments resulted in significantly different rates of forgetting. The group of students who were given statements of each activity's objective showed a positive gain in performance over time. In addition, the differences in the over-all performance scores resulting from the four treatments were not identically reflected at each ability level.

CONCLUSIONS

The conclusions based upon the findings yielded by statistical treatment of the data are presented below. Each conclusion is stated in terms of the relevant research hypothesis.

1. The hypothesis that giving students statements of the behavioral objectives (with examples) before each activity of an instructional unit results in higher achievement scores for the students so informed than those students who are given no information beyond the actual instructional unit activity is not supported by the data.

2. The hypothesis that giving students copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit results in higher achievement scores for the students so informed than for those students who are not given any information beyond the actual instructional unit activity is not supported by the data.

3. The hypothesis that giving students statements of the behavioral objectives (with examples) and copies of the learning sequence before each activity of an instructional unit results in higher achievement scores for the students so informed than for those who are not given any

information beyond the actual instructional unit activity is not supported by the data.

4. The hypothesis that giving students statements of the behavioral objectives (with examples) and copies of the learning hierarchy before each activity of an instruction unit results in higher achievement scores for students so informed than for students who are given only statements of the behavioral objectives (with examples) before each activity of an instructional unit is not supported by the data.

5. The hypothesis that giving students statements of the behavioral objectives (with examples) and the learning hierarchy before each activity of an instructional unit results in higher achievement scores for students so informed than for students who are given only copies of the learning hierarchy with examples of sequence cells at the beginning and at the end of an instructional unit is not supported by the data.

6. The hypothesis that the differences in achievement scores resulting from the four treatments are not identically reflected at each ability level is not supported by the data.

7. The hypothesis that the four treatments have differential effects on the over-all performance of students

is not supported by the data.

8. The hypothesis that the four treatments result in different rates of forgetting is supported by the data.

9. The hypothesis that the differences in over-all performance scores resulting from the four treatments are not identically reflected at each ability level is supported by the data.

DISCUSSION

The conclusions of this study may be conveniently partitioned into two areas:

1) The effects of the treatments on immediate achievement scores obtained on an immediate posttest, and
2) The effects of the treatments on over-all performance scores and rate of forgetting as measured by an immediate posttest and a delayed posttest.

This section will discuss the conclusions of this study in the perspective of these two separate areas.

Immediate Achievement

The results of the study appear to be quite conclusive in terms of there being no observable differential effects of the four treatments on immediate achievement scores. The analysis of the data provided no support for qualifying this finding on the basis of ability levels. There are several possible reasons that these results were obtained:

1) Complexity of study. The unfamiliar format of additional instructions concerning the behavioral objectives and the learning hierarchy may have been a confounding factor in the study. The degree of unfamiliarity with the overall format was exemplified on a few occasions in the beginning of the study when students who had received examples of all the subtasks in the hierarchy were observed trying to work them rather than note the solutions provided. The level of complexity may have been heightened by those students whose reading abilities were not adequate for the material.

2) Format of instruction. The fact that the instructional format was self-instructional in a single medium may have been a confounding factor. One might question whether a similar study conducted in a typical classroom situation without the use of self-instructional materials would yield data that would reflect differential effects of the four treatments on immediate achievement scores.

3) Administration procedures. Another factor which may have affected the results of this study is the procedure used to administer the treatments. The author administered

the four treatments within each of four classroom. The individual booklet containing each activity was assumed to sufficiently insulate each student from the other three treatments. However, the fact that the students were aware that they were receiving four different sets of instructional materials may have sufficiently aroused their curiosity to lead to the sharing of information outside of the classroom. The investigator became aware of one situation where students were sharing notes outside of class.

4) Validity of the learning hierarchy. The validity of the hierarchy is reflected by the range of the final validation analysis ratios (.66 to 1.00) shown in Figure 5, page 76. Although a few of the ratios did not meet the criteria selected by the author, the range of the validation ratios indicates that the hierarchy has considerable validity.

Over-all Performance and Rate of Forgetting

The results of the study indicate that if the levels of ability of students are not considered, then the benefits in terms of over-all performance to be derived from either of the four treatments do not differ significantly at the 0.05 level. However, the presence of the significant treatments-by-levels interaction effect found in the study indicate

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that differences between the effects of the four treatments do exist within the three levels of ability. Therefore, any predictive statement concerning the effects of the four treatments on over-all performance must be qualified by specifying the levels of ability involved.

The observed treatments-by-levels profiles reveal that giving students statements of the behavioral objectives of an activity accompanied by a copy of the learning hierarchy is more beneficial (when compared to the other three treatments) to the students in the middle ability level than to those students in either the high or the low ability levels. It is reasonable to argue from this finding that the low ability level students were not able to assimilate the additional information and, thus, received no benefit from being given the objectives and/or the learning hierarchy. From the treatments-by-levels profiles, one might project that the additional information about objectives actually confused the low ability level students. The students in the high ability level group apparently received no differential benefit from either of the treatments. One might argue in this case that the pre-organizing benefit possibly provided by the statements of the behavioral objectives and/or learning hierarchy were not needed. Perhaps the high level students were able to infer the objectives and the learning sequence itself. In the case of the middle ability group, one could assert that the objective statements accompanied by the learning hierarchy enabled them to gain a helpful perspective about the mathematical skill which they were expected to learn.

The significant treatment-by-posttest interaction effects found by the repeated measure analysis is a second indication that differences in the four treatments do exist. This interaction effect reveals that the treatments result in different rates of forgetting. The more striking difference observed is that while the rate of forgetting of treatments T_1 and T_3 show a negative gain and the rate of forgetting of treatment T_4 shows neither gain nor loss of the mathematical skill, the slope of the treatment T_3 retention curve shows a positive gain. The findings of this study reveal that while giving students statements of the behavioral objectives of each activity of the instructional unit did not result in a relatively high immediate achievement scores, the treatment did result in greater resistance to forgetting than the other methods of instruction. One could reasonably argue that the students in treatment T_2 were able to gain in ability to perform the terminal task because the combination of being aware of the behavioral objectives and the two-week period between posttests resulted in their being able to better assimilate the instructional material they had received.

VALIDITY OF THE STUDY

Two general criteria of a well-planned research design were considered by the investigator: internal and external validity. Consideration is given in this section to the precautions taken in this study to control extraneous variables and the generalizability of the findings is discussed.

Internal Validity

The design of the study included the following list of precautions which were taken to assure that any possible treatment differences were indeed the result of treatment conditions.

1. Concurrent events. All students in the study were enrolled in Math 205 at Towson State College. The full time teachers of each of the four sections of Math 205 used in this study had followed the same syllabus for the course. No portion of the mathematics topic chosen for this study had been taught in Math 205 prior to the time of the study. The students in the available population were enrolled in only one mathematics course. The full time teachers did not

teach any topic during the two-week period between immediate and delayed posttests that required the mathematical skills learned during the experiment as a foundation. In addition, the full time teachers did not conduct a review of the mathematics skills learned during the experiment. Hence, it was felt that the events occurring before the experiment, during the three-week instructional period, and during the two-week period between the two posttests were sufficiently controlled for the purposes of the study.

2. Maturation. The effects of maturation refer to general processes operating within individuals as a result of passage of time, including growing older, growing hungrier, or becoming more fatigued. Although there was some time pressure accompanying the learning experiences, any change in productivity due to fatigue or pressure was spread evenly over the four groups. The fact that the students were randomly assigned to treatments also reduced the possible threat due to maturation. Because slow students were allowed to attend additional class sessions, it is considered unlikely that time had any influence on the results of the experiment.

3. Testing. The same test was used for the delayed posttest as was used for the immediate posttest. The existence of a carry-over effect is a possibility. In order to

reduce the possibility of such an effect, each student received a different version of the performance test for the delayed posttest than he received for the immediate posttest. Since the three versions of the performance test consisted only of three different orders in which the problems were listed, this precautionary step against the carryover effect may not have been sufficient.

4. Criteria instrumentation. Any bias or subjectivity in scoring the posttests was eliminated by the objective nature of the tests themselves, for which a single scoring key was used for both immediate and delayed posttest. Grading of the posttests was done by grading the same problem for all eighty-eight students before grading the next problem. Thus, any possible subjectivity in grading was greatly reduced. Reliability of the performance test under each treatment was determined and considered very adequate. Therefore, any effects due to criteria instrumentation do not pose a significant threat to validity.

5. Selection. Since students were randomly assigned, after being blocked according to ability levels, to treatment groups and all students were enrolled in the same course, differential effects resulting from selection were felt to be absent.

6. Experimental mortality. One student in the low

ability group assigned to the control group withdrew from the course before completion of the experiment. Four other students were absent from class the day the unannounced delayed posttest was administered. Thus, one score on the immediate posttest and five scores on the delayed posttest were lost. Since the percentage of lost scores was very small, it was considered adequate to substitute the mean of the cell of the missing data for the missing score. It was not felt that any influence due to this mortality was sufficient to mask treatment effects.

7. Chance. The possibility that the findings reflected chance rather than true effects was minimized by selection of the 0.05 level of significance. The possibility of reflecting chance was further reduced by the decision to use the experiment as the base of $\alpha = 0.05$.

8. Independent treatments. One serious challenge to the findings of this study is that the students may not have been independent of the different treatment materials being used in the same classroom. Since the students knew they were using four different sets of material, it is highly likely that they discussed the differences outside of class. The curious student could easily have become aware of the learning sequence and the terminal task from inquiring outside of class of a friend who had received a copy of the

learning hierarchy in his treatment materials. As a precaution of any materials leaving the classroom, each student was checked off a class list as he returned his material at the end of class. However, the possibility of the treatments not being independent remains the most outstanding threat to this study.

External Validity

The target population of this study consisted of the set of all elementary education college students who enroll in the second semester of a two-semester sequence of college mathematics courses designed to teach the content material required for elementary school teachers. The available population was a subset of the target population. The sample was in like manner a subset of the available population. Hence, the results of this experiment can be generalized to the target population. Replication of the experiment in other target population is needed to increase the generalization of the findings. The fact that the students in the study appeared to be highly motivated may have been a threat to the findings and to generalization to the target population. However, the level of motivation, if it did exist, seemed generally distributed among the students in the study and, hence, may not be a true threat to the external validity of the study.

IMPLICATIONS

The implication derived from the findings of this study pertain to current practice and future research.

Implications for Current Practice

1. The findings of this study lend no support to the assertion that telling students the behavioral objectives and/or the learning hierarchy of a unit of instruction will increase their performance on immediate achievement tests.

2. The stating of behavioral objectives in textbook format without explanations by the teacher do not significantly help the student to perform higher on an immediate achievement test. There is some basis to believe that such information may even confuse the student.

3. The procedure of informing the student of the behavioral objective may be an effective method for providing resistance to forgetting of learned skills.

4. For the student of middle ability level, being informed of behavioral objectives and the learning hierarchy seems to be a beneficial method for improving his over-all performance.

Implications for Future Research

The results of this study have the following implications for research involving behavioral objectives and learning hierarchies.

1. The use of self-instructional materials should be discouraged as a means for removing the teacher factor from future experiments.

2. Future studies might more effectively be conducted if the possibility of sharing of treatments outside of class were minimized.

3. Studies should be conducted in a typical classroom situation without the use of programmed materials.

4. Studies should be conducted in other academic areas and in other academic levels (elementary, secondary, and college).

5. Future experiments should consider the use of ability levels as a blocking factor.

6. The findings indicate that more long-term studies should be conducted with perhaps more than two posttests.

USE TO BE MADE OF FINDINGS

1. The findings of this study will be presented in a preconference seminar at the 25th annual conference of the Association for Supervision and Curriculum Development in San Francisco, California on March 12-13, 1970. The seminar is entitled Systems Analysis and Curriculum Development. The author will be a member of a team from the University of Maryland Baltimore County who will plan and conduct the seminar.

2. The author, in conjunction with Dr. Henry Walbesser and others, has submitted an application to the American Educational Research Association to conduct a symposium at the 1970 Annual Meeting of AERA in Minneapolis, Minnesota, March 2-6, 1970. The findings of this study will be discussed in the symposium as well as the findings of other recent research on the use of behavioral objectives in teaching.

3. The findings of this research conducted with college students will provide the basis for an extension of the study to high school students without the use of programmed materials.

4. An article discussing the findings has been accepted for publication in the 1969 Fall issue of the journal for the Maryland Association for Supervision and Curriculum Development.

5. A more detailed article, providing tables and graphs of data is under draft. This article will be submitted for publication in Educational Leadership, the journal of the national organization of the Association for Supervision and Curriculum Development.

6. The findings are also being used by the author in the training of in-service teachers.

APPENDIX A

INTRODUCTION AND TEXT MATERIAL

(CONTROL TREATMENT T_1)

INTRODUCTION

The materials for your next unit of instruction were written by the University of Maryland Mathematics Project. Perhaps you will find that the format of the materials will differ from the textbooks that you have previously used. Ordinary textbooks usually require homework. Homework is not required for this material. In this material, all your work will be done individually in class. You will be asked to answer questions by writing on the blanks provided throughout the materials or by performing the other tasks requested.

These materials are designed around what you do. If you work on each task at the point it is called for, you will find that the materials have a completeness about them that is not in an ordinary textbook. You will also find that your response at each step will help you to acquire the skills upon which you will be tested. FOR YOUR BENEFIT do not read further until you have completed each task the way you think it should be done. (No one is going to count the number of questions answered correctly--the tasks are for your benefit.) If you answer incorrectly or perform a task incorrectly, re-read the questions and correct your original response.

At the beginning of each class, your instructor will inform you of the maximum number of lessons which you will be permitted to complete that day. Upon completing a lesson, raise your hand and you will be given another lesson. When you have completed all the lessons permitted for that day, raise your hand and you will be given your "check-up questions." Upon completing your "check-up questions," raise your hand and the instructor will pick up your questions. If you complete the check-up questions before the end of the class period, you may work on some reading assignment. Although everyone will be working at his own pace, it is suggested that you take your time.

You are expected to make arrangements with your instructor for any lesson you might miss because of absence.

Here are some examples to show you how the format will look.

Sets of elements are denoted by the two braces $\{$ and $\}$. Given the two sets of integers

$$\{1, 2, 3, 4, 5\} \quad \text{and} \quad \{4, 6, 7\},$$

circle in red that integer which is common to both sets. Do not read further until you have performed the task!

Did you circle the 5? You didn't? You are correct, the 5 is not in both sets. However, the 4 is in both sets and you are correct if you circled the 4 in both sets.

Given the following two sets

$$\{1, 2, 3\} \quad \text{and} \quad \{4, 5\} ,$$

list all the integers which are in either of the two sets: __________. Have you listed the integers? Do it now before they disappear--or before you read further.

Did you list only the integers 1, 2, and 3? If you did, re-read the question. You were asked to list the integers in either of the two sets. The integers which are in either of the two sets are 1, 2, 3, 4, and 5.

Turn the page. You are ready to begin.

I. Do You Remember Sets?

You probably remember many things about sets and the union and intersection of sets.

Do you remember that a set consisting of the integers -2,0,1,3, and 4 can be shown like this?

$$\{-2, 0, 1, 3, 4\}$$

You do? Great! How would you show a set containing integers 6,7,8, and 9? Write it in this space: ________________ Stop! Do not read any further until you have written the set in the blank.

Great! That's the way we'll be doing it throughout this material. Perform the task requested before reading further. Now, the way to write the set of integers mentioned is

$$\{6, 7, 8, 9\}.$$

Definition 1: The union of two sets consists of all the distinct elements in either of the two sets.

Suppose we have the two following sets

$$\{1, 2, 3, 4\} \quad \text{and} \quad \{5, 6\}.$$

The union of these two sets will therefore be the set

$$\{1, 2, 3, 4, 5, 6\}.$$

One way of showing this is

$$\{1, 2, 3, 4\} \ \mathrm{U} \ \{5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

The symbol U denotes union.

Alright, let's see what you would do with this:

$$\{1, 4, 5, 6\} \ \mathrm{U} \ \{8, 9\} = \underline{\hspace{5cm}}?$$

Go ahead, complete the blank.

The response was correct if you wrote $\{1, 4, 5, 6, 8, 9\}$. That is,

$$\{1, 4, 5, 6\} \ \mathrm{U} \ \{8, 9\} = \{1, 4, 5, 6, 8, 9\}.$$

Got it? O.K., let's move on.

What is the union of the next two sets? Write their union in the blank:

$$\{1, 2, 3, 4\} \ \mathrm{U} \ \{4, 5, 6\} = \underline{\hspace{5cm}}.$$

Don't read further until you fill in the blank!

Did you write $\{1, 2, 3, 4, 4, 5, 6\}$?

You did? Go back to Definition 1. Notice that the union of two sets consists of all the distinct integers which are in either of the two sets. This means if you have an integer that is in both sets you include it in the union only once. Therefore, the correct response is $\{1, 2, 3, 4, 5, 6\}$, counting the integer 4 only once. It is important to note that the order of the integers in the set is not important. That is, $\{1, 2, 3, 4, 5, 6\}$ is the same set as $\{4, 3, 6, 1, 5, 2\}$.

Suppose you had the same two sets, $\{1, 2, 3, 4\}$ and $\{4, 5, 6\}$, and you were asked to name the integer that is common to both sets. The integer you would name is _____.

Definition 2: The intersection of two sets is that set of elements which are common to both of the two original sets. The symbol $\cap$ denotes intersection. Hence

$$\{1, 3, 5, 9\} \cap \{5, 7, 8\} = \_\_\_\_\_\_\_\_?$$

Do not read further until you have completed the blanks.

Right, if you said the integer 4. It is in the set $\{1, 2, 3, 4\}$ and in the set $\{4, 5, 6\}$. Did you write $\{5\}$ for the intersection of $\{1, 3, 5, 9\}$ and $\{5, 7, 8\}$? $\{5\}$ is the correct response. $\{5\}$ is used rather than just 5 because the intersection of two sets is also a set.

It is possible for a set to exist without any members. Such a set is called an empty set and is denoted by $\{\ \}$ or $\emptyset$. Another name that is given to such a set is the null set.

What is the intersection of the next two sets?

$$\{1, 3, 5\} \cap \{6, 7\} = \_\_\_\_\_\_\_\_?$$

Did you write $\emptyset$ or $\{\ \}$? You're with it! There is not an integer in the two sets that is common to both sets. Therefore, the $\cap$ set is empty.

Just to make sure that you're clear on the idea of the $\cup$ and the $\cap$, let's consider the set $I = \{1, 2, 3, 4, 5, 6, 7, 8\}$. With $I = \{1, 2, 3, 4, 5, 6, 7, 8\}$ let

$A = \{1, 2, 3, 5, 7\}$
$B = \{1, 3, 6, 7\}$
$C = \{2, 4, 6, 8\}$
$D = \{3, 6\}$
$E = \{2, 4, 5, 6, 8\}$
$F = \{2, 5, 7, 8\}$

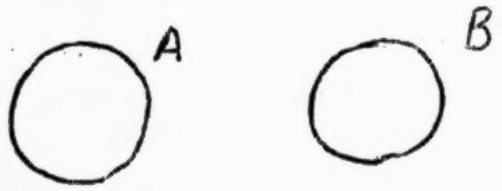
Note that $A \cap B = \{1, 2, 3, 5, 7\} \cap \{1, 3, 6, 7\} = \{1, 3, 7\}$ and that $C \cap D = \{6\}$.

Complete the following sets:

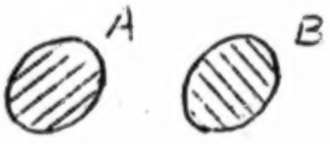
1. $A \cap C =$ ____________________
2. $B \cup D =$ ____________________
3. $D \cap A =$ ____________________
4. $C \cap F =$ ____________________
5. $(C \cap F) \cap E =$ ____________________

Problem No. 5 is particularly interesting. In No. 4 you found the set $C \cap F$. No. 5 asks the question: What is the $\cap$ of the set $C \cap F$ with E? Answers to the five problems are shown on page 7.

A Venn diagram is not unfamiliar to you. Let's look at two Venn diagrams representing two sets A and B which have no elements in common.

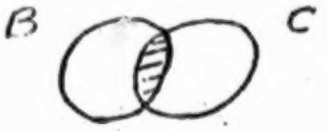


The set A ∪ B consists of all the elements in set A and set B. We can represent all the elements in each set by shading each set.

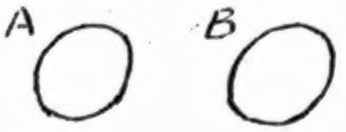
Thus A ∪ B = 

The intersection of two sets can also be shown with Venn diagrams.

Consider two sets B and C which have some elements in common. This condition may be represented by

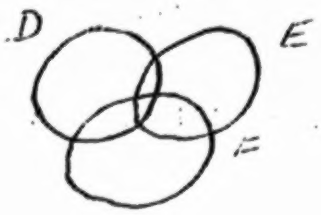


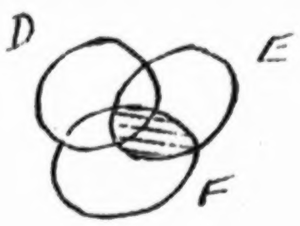
with the shaded region representing the ∩. Again, the ∩ of



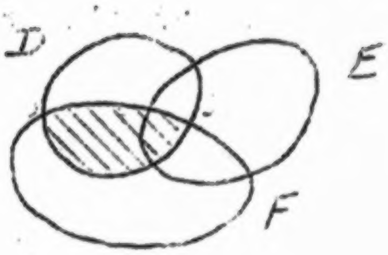
is the empty set. That is, A ∩ B = { }. This is true because sets A and B have no elements in common.

Given three sets D, E, and F

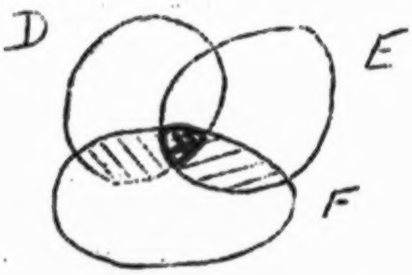


E ∩ F can be shown as 

D ∩ F can be shown as

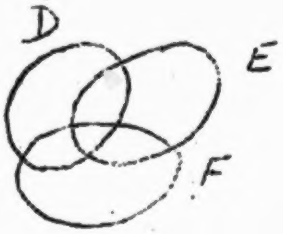


Again (E ∩ F) ∩ (D ∩ F) is shown as the cross-hatched region in red:

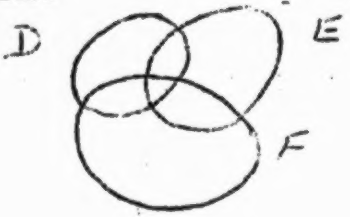


Now, you can perform the following tasks:

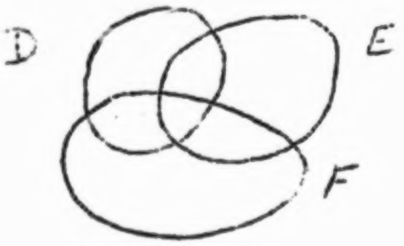
1. Shade D ∩ E in:



2. Shade E ∩ F in:



3. Having completed the above two tasks, shade (D ∩ E) ∩ (E ∩ F) in:



The correct shading of these three tasks is shown on page 7 .

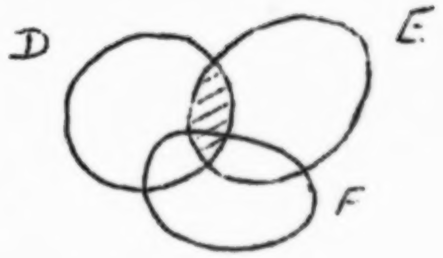
Answers to Performance Tasks

Page 4.

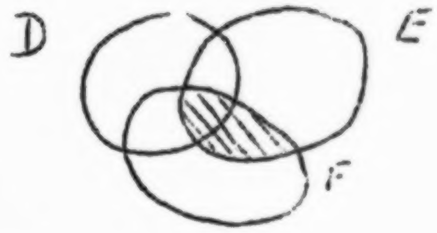
1. $\{2\}$
2. $\{1, 3, 6, 7\}$
3. $\{3\}$
4. $\{2, 8\}$
5. $\{2, 8\}$

Page 6.

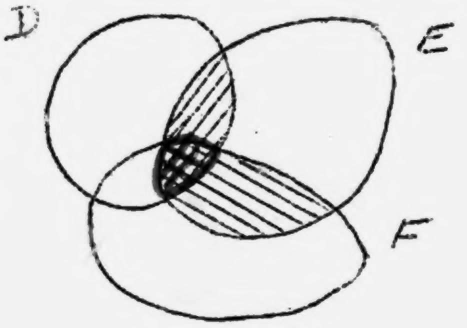
1. $D \cap E$



2. $E \cap F$



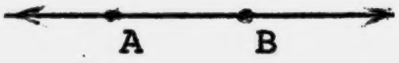
3. $(D \cap E) \cap (E \cap F)$



II. The Line

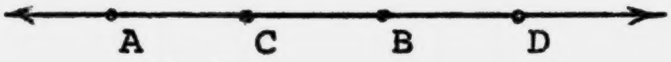
A line can be drawn through any two points. Thus if we have a line, *any* two points on the line can be used to refer to the line--that is, to name the line.

Hence the line



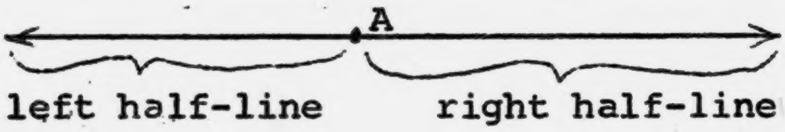
can be named line AB. Since a line extends indefinitely in both directions, we place arrows on the line to indicate this property. Thus we can denote line AB by the symbol $\overleftrightarrow{AB}$.

A line contains many more than two points. Since we have noted that a line can be noted by *any* two points on the line, the line



can be named $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, or $\overleftrightarrow{CD}$ *or any combination of two points on the line*.

Let us partition a line into three parts by denoting a point A on the line.

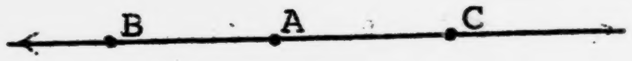


Definition 3: The portion of the line to the right of A is named the *right half-line* and the portion to the left is named the *left half-line*. The end-point A of both half-lines *is not included as part* of either half-line.

Thus, the entire line can be expressed as

left half-line U pt. A U right half-line

If we were to denote points B and C on two opposite half-lines (having A as their end-point) as

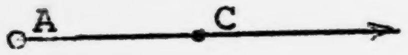


we can express the entire line as

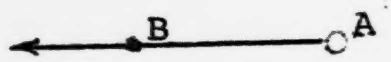
half-line AB U A U half-line AC.

Notice that the half-line is denoted by placing the end-point A *first* in its order of letters. Also, notice that point A is denoted by writing A.

Perhaps it would be helpful to show the half-line AC as follows

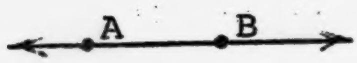


The *open circle* at A denotes that point A *is not* included. The *solid point* at C denotes that C *is* included. Thus half-line AB may be shown as



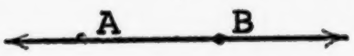
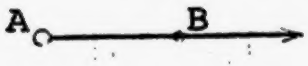
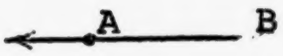
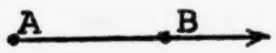
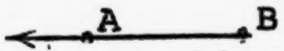
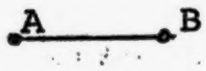
Definition 4: The set of points consisting of a half-line and its end-point is called a *ray*. The end-point of the half-line is the *end-point* of the ray.

As there is one and only one line containing any two points A and B,



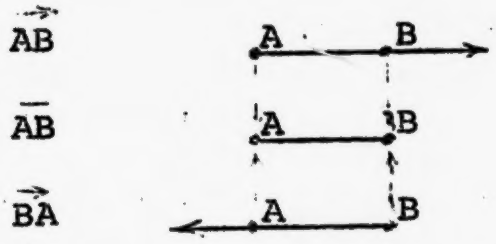
there is one and only one ray with end-point A and containing the point B. We call this the ray AB and write it as $\overrightarrow{AB}$. There is also the ray $\overrightarrow{BA}$ with end-point B and containing A.

We have

$\overleftrightarrow{AB}$		Line AB
		Half-line AB
		Half-line BA
$\overrightarrow{AB}$		Ray AB
BA		Ray BA
AB		Line segment AB

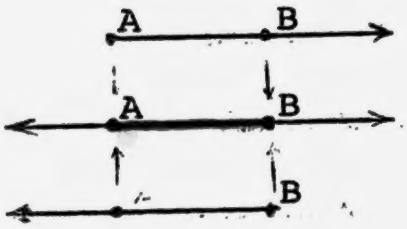
Definition 5: The line segment AB, written as $\overline{AB}$, consists of the points which $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have in common. That is, the set of points $\overline{AB}$ is the ____________ of the set of points $\overrightarrow{AB}$ and the set of points $\overrightarrow{BA}$. Did you write the word intersection in the blank? Remember, the $\cap$ of two sets on a line is the set of points on the line which are common to the two sets.

Look at what's going on:



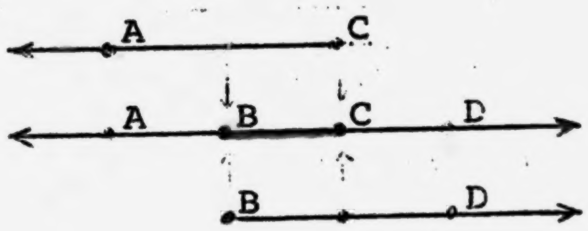
The points A and B and all of the line in between A and B are found in both rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Therefore, their $\cap$ consist of points A and B and all of the line between points A and B. This is exactly $\overline{AB}$, by Definition 5.

In order to make it easier to picture, let's show the set of points $\overrightarrow{AB}$ above line $\overleftrightarrow{AB}$ and the set of points $\overrightarrow{BA}$ below the line $\overleftrightarrow{AB}$. The intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ on the line $\overleftrightarrow{AB}$ is $\overline{AB}$ shown in red.



Suppose we had a line with four points represented on it.

The $\cap$ of $\overrightarrow{CA}$ and $\overrightarrow{BD}$ are determined by representing one set of points above $\overleftrightarrow{AD}$ and the other set of points below $\overleftrightarrow{AD}$ as:



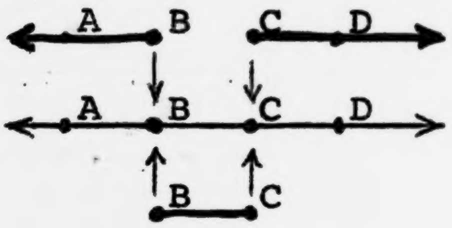
The $\cap$ of $\overrightarrow{CA}$ and $\overrightarrow{BD}$ is shown in red. That is,

$\overrightarrow{CA} \cap \overrightarrow{BD} =$ ________________.

Did you respond by writing the answer in the blank? Don't go further until you take the time to write in the answer. You will have more like this one.

What did you write? $\overline{BC}$ consists of the set of points which are common to both $\overrightarrow{CA}$ and $\overrightarrow{BD}$ and therefore $\overline{BC}$ is the correct answer.

Suppose we consider the set of points consisting of $(\overrightarrow{BA} \cup \overrightarrow{CD})$ as one set (shown in blue) and the set of points $\overline{BC}$ as a second set (shown in green). Their $\cap$ could be found in a manner similar to the preceding examples:



Notice that the points between B and C on $\overline{BC}$ are not found in the set $\overrightarrow{BA} \cup \overrightarrow{CD}$. However, the points B and C are common to the set $\overrightarrow{BA} \cup \overrightarrow{CD}$ and to the set $\overline{BC}$.

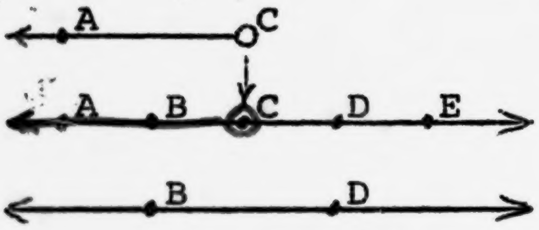
That is,

$$(\overrightarrow{BA} \cup \overrightarrow{CD}) \cap \overline{BC} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Be sure and fill in the blank by looking at the set of points shown in red. What did you answer? The correct response is B U C.

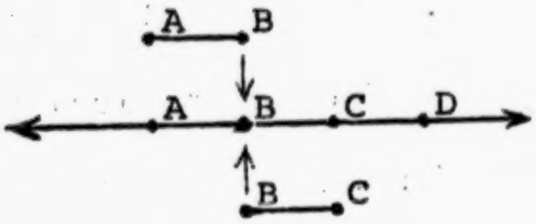
You will now have a chance to name the $\cap$ of some sets by observing their intersection on a line. In all cases the points shown above the line represent one set of points on the line and the points shown below the line represent a second set of points on the line.

1.



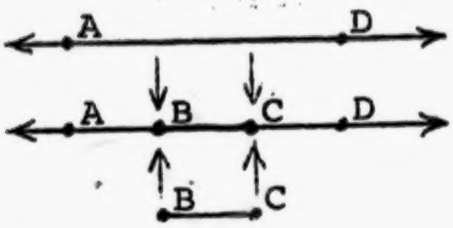
half-line CA $\cap$ $\overleftrightarrow{BD}$ = ____________.

2.



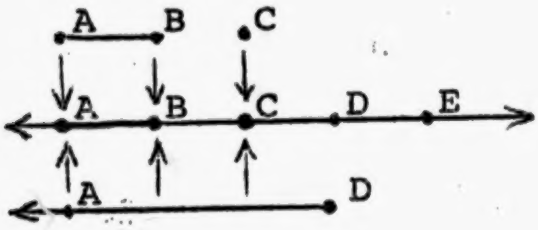
$\overline{AB} \cap \overline{BC} =$ ______________________________.

3.



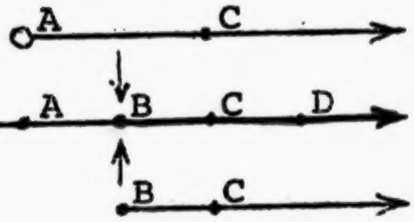
$\overleftrightarrow{AD} \cap \overline{BC} =$ ______________________________.

4.



$(\overline{AB} \cup C) \cap \overrightarrow{DA} =$ ______________________________.

5.



half-line AC $\cap \overrightarrow{BC} =$ ______________________________.

Have you completed the above performance tasks? No? Take the time to fill in the blanks. The practice now will help you later.

You have finished the tasks? Check your answers with these: (1) half-line CA; (2) B; (3) $\overline{BC}$; (4) $\overline{AB} \cup C$; (5) $\overrightarrow{BC}$.

III. The Line Re-visited

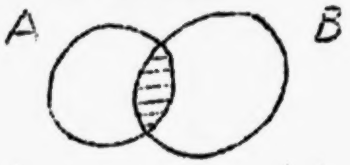
In Part II you were shown what the intersection of two sets of points on a line was. It always consisted of another set of points. What do you do to determine the intersection set?

When we are speaking of the intersection of two sets we are speaking of those points which are __________ to both sets. When we were working with Venn diagrams, we __________ the regions which were common to both sets. Stop! Stop! Don't read on! --That is, until you write in the missing words in the blanks.

Now! Did you write in the words common and shaded, respectively, in the above blanks? You did? Then you may skip to the next paragraph right now. If you did not, perhaps you have forgotten that the intersection of these two sets

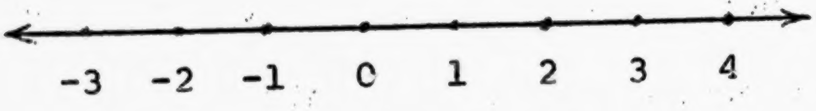
$$\{1, 2, 3, 4\} \quad \text{and} \quad \{4, 5\}$$

is the set $\{4\}$ since 4 is common to both sets. Also, the intersection of sets A and B is shown



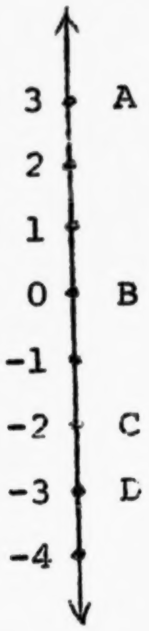
by the shaded region.

We can find the intersection of two sets of points on a line and indicate the intersection set in a way very similar to that which we've already been using. To show what I mean by this, let's consider a number line on which the integers are associated with particular points of the line:



Definition 6: The numbers . . . -3, -2, -1, 0, 1, 2, 3, 4, . . . are called the coordinates of the points associated with them on the line.

The number line does not have to be drawn in a horizontal position. Let us call the number line pictured below AD.



The coordinate of point A is +3, of point B, 0, of point C, -2, of point D, ____. The blank is for you to fill in! Remember that we indicate that the line extends indefinitely in each of two directions by placing an arrow at each end of our figure.

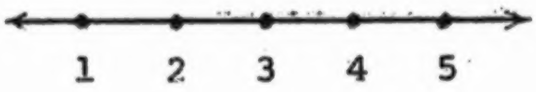
How do you describe sets of points on a line? We describe sets of points on a line by constructing a graph of the points on a line.

Definition 7: The set of points associated with a set of numbers is called the graph of the set.

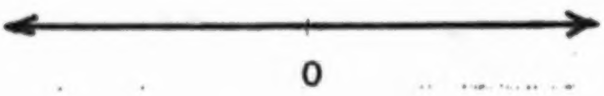
Let's construct the graph for the set A, where

$$A = \{1, 2, 3, 4, 5\} .$$

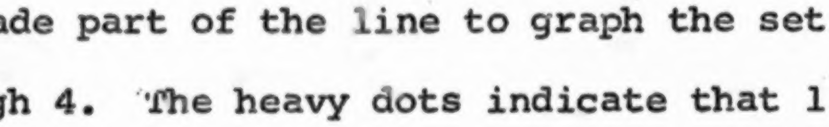
We place solid dots on a number line at each of the points which correspond to a number of the set A, as follows:



Suppose another set consisted of all the real numbers on a line, rather than just integers. Since the number line represents all real numbers, the graph of the set of real numbers would be the entire line. Thus, the graph would show a heavily shaded line rather than a series of dots.

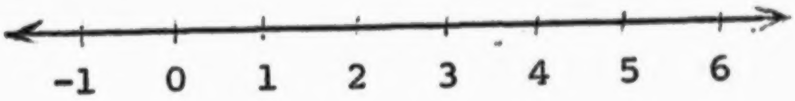


We shade part of the line to graph the set of all numbers 1 through 4. The heavy dots indicate that 1 and 4 are members of the set.



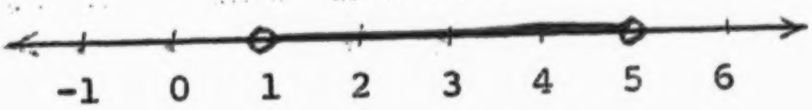
Notice that what we have shown is a line segment. Using a red pencil, why don't you construct the graph of all the numbers 2 through 5 on the line below:

1.

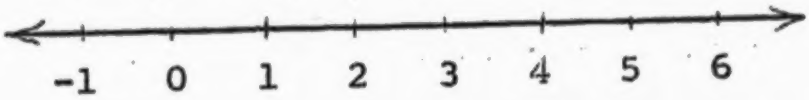


Great, you'll move faster if you take time to perform each task. Did you shade the part of the line from 2 through 5 with heavy dots at 2 and at 5? Check your graph with the one on the answer sheet, page 15.

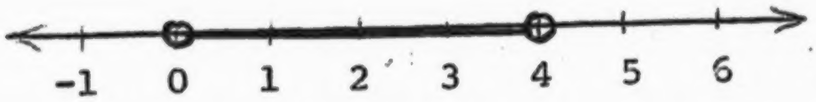
We use hollow dots when points are not included. For example, we graph the set of all numbers between 1 and 5 as in the following figure:



It's your turn again. Using a red pencil, construct the graph of all the numbers between 0 and 4 on the line below:

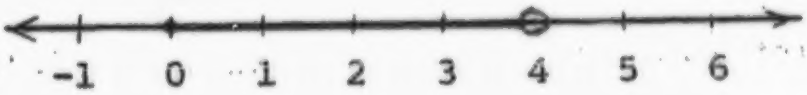


Did you use a red pencil? It will help. Your graph would look like

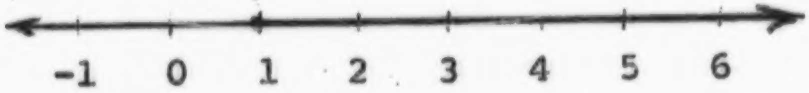


except yours will show the shaded portion and the open dots in red. Since you were asked to construct the graph of all numbers between 0 and 4, the points at 0 and at 4 are not included. Thus, be sure that you show open dots at 0 and at 4.

What would our graph look like if we wanted to show the set of points which include 0 and all the numbers between 0 and 4? You're right! We would show a heavy dot at 0 and an open dot at 4 with the line shaded in between 0 and 4. That is, it would look as follows:

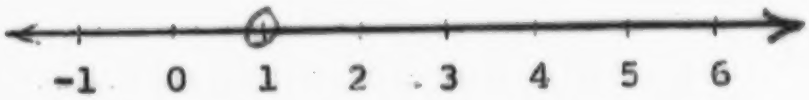


Next let's construct the graph of all numbers greater than or equal to 1. We place a heavy dot at 1 and draw an arrow to show that all the numbers greater than 1 are numbers of the set, as follows:



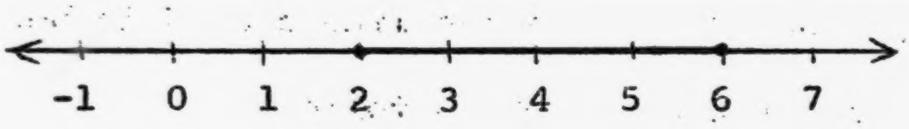
You will recognize this graph as a ray.

Now, the graph of the set of all numbers greater than 1 would be as follows:

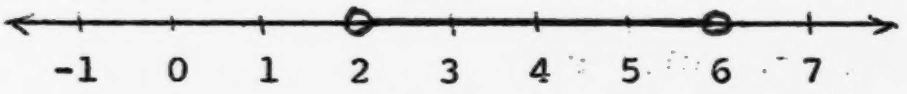


Here we placed a hollow dot at 1 to show that it is not to be included in the graph and we drew an arrow to the right to show that all numbers greater than 1 are members of the set. This, of course, is a half-line.

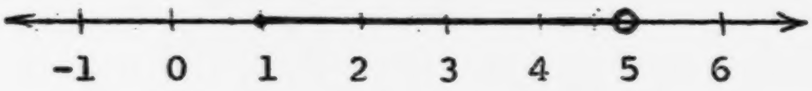
Sometimes the symbolism [a, b] is used to represent a set of points on a line in the interval from a to b, with both end-points a and b included in the set. For an interval of real numbers, [2, 6] represents all of the real numbers from 2 through 6 inclusive. The graph of the interval [2, 6] on the number line is



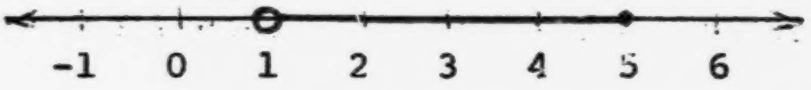
We use the symbolism (a, b) to represent the points on the interval from a to b, exclusive of these two end-points. The graph of the interval (2, 6) on the number line is



A combination of these symbols may be used to indicate that one end-point is included but not the other. Thus [a, b) is used to represent the end-points in the interval from a to b, including a but not including b. The graph of [1, 5) is

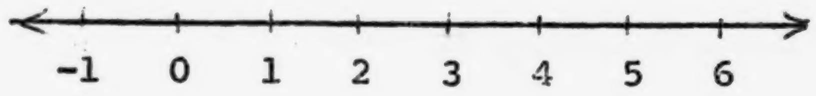


while the graph of (1, 5] is

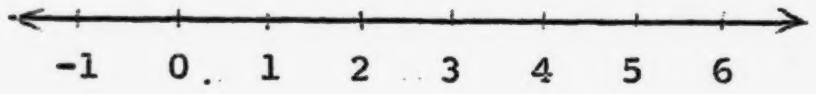


O.K., ready to try some? On the line beside each stated interval, construct its graph

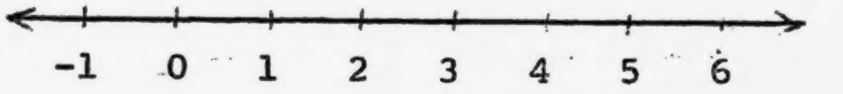
2. [1, 3]



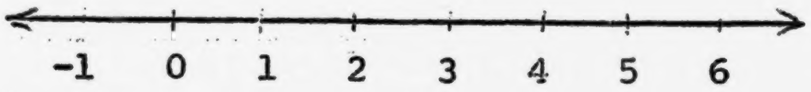
3. (-1, 4)



4. (-1, 2]



5. [0, 6)

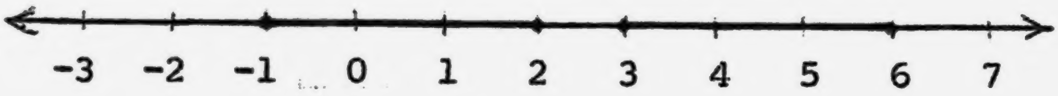


The graphs of each interval are shown on page 15.

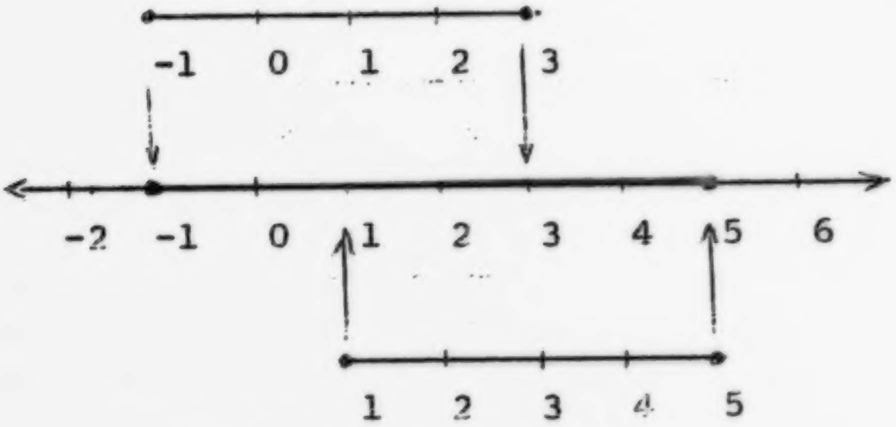
The union of two intervals, say

$$[-1, 2] \cup [3, 6]$$

may be shown as

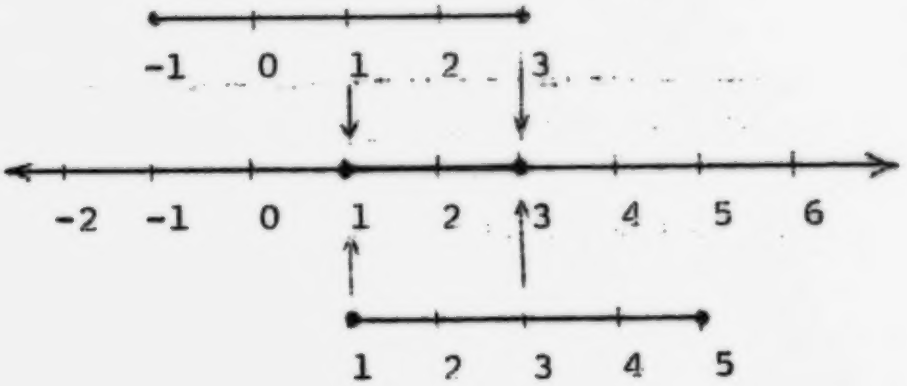

-3
-2
-1
0
1
2
3
4
5
6
7

The construction of the union [-1, 3] and [1, 5] is a little more difficult to show on paper. Wait a minute. Let's use the procedure for constructing the $\cup$ or $\cap$ of rays and line segments that we used earlier. Let's show one set above the line and one set below the line and project the union of the two sets on the line. Hence, we have for [-1, 3] U [1, 5]


-1
0
1
2
3
-2
-1
0
1
2
3
4
5
6
1
2
3
4
5

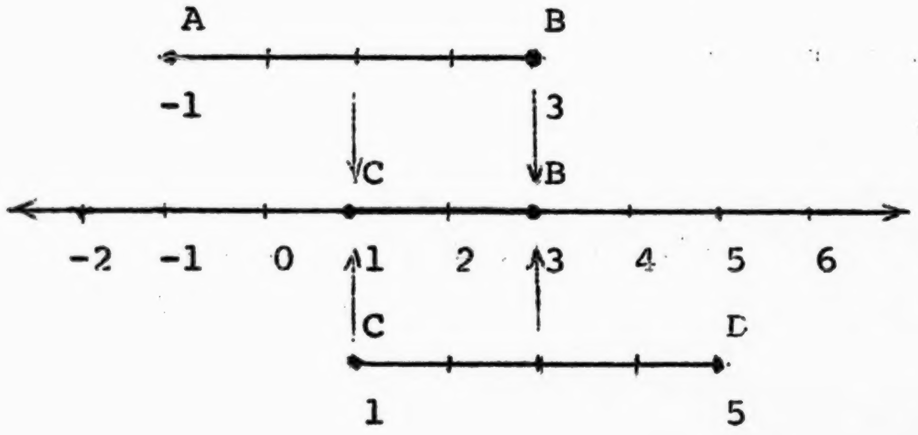
where [-1, 3] U [1, 5] is shown in red on the line.

The construction of the intersection of [-1, 3] and [-1, 5] is shown in a similar manner. The actual intersection set is projected onto the line. Thus, the graph of [-1, 3] ∩ [1, 5] is


-1
0
1
2
3
-2
-1
0
1
2
3
4
5
6
1
2
3
4
5

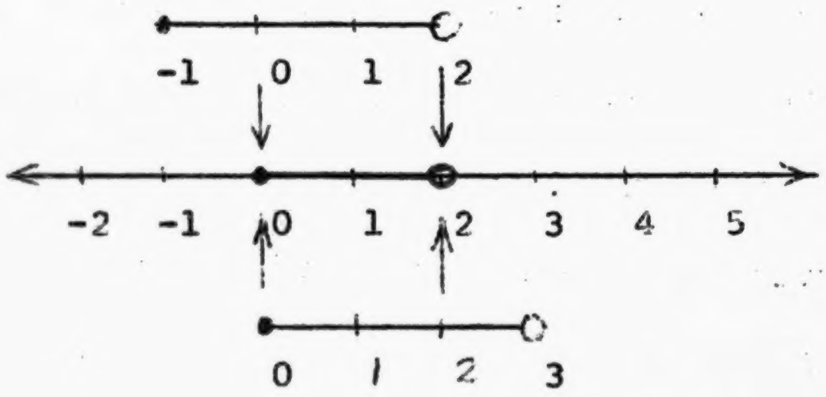
That is, [-1, 3] ∩ [1, 5] = [1, 3]. Notice that since 1 and 3 are common to both sets, 1 and 3 are included in the ∩ set.

These intervals are line segments. If we were to name them as such we would have



That is, $\overline{AB} \cap \overline{CD} = \overline{CB}$

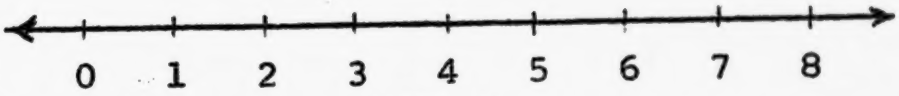
What about $[-1, 2) \cap [0, 3)$? Let's construct its graph. We'll show the set [-1, 2) above the line and [0, 3) below the line. Projecting



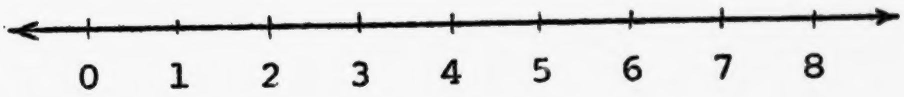
their $\cap$ onto the line, we note that 0 is included in both [-1, 2] and [0, 3), while 2 is included in [0, 3) but not in [-1, 2). Thus, a heavy dot is shown at 0 and an open dot is shown at 2. Since all the real numbers between 0 and 2 are included in both sets, the line is shaded red in between 0 and 2.

In each of the following cases, construct the U or ∩ as requested. Show the U set or the ∩ set in red. You'll find help on these on page 15. Watch our for the symbols [and) .

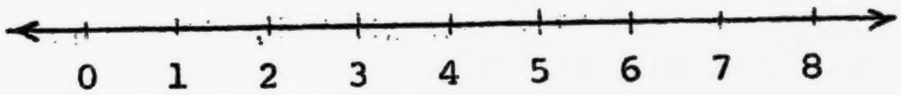
6. $[2, 5) \cap [3, 7]$



7. $(2, 5] \cup [3, 7)$



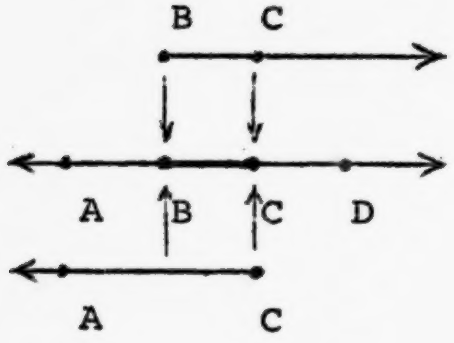
8. $[0, 3] \cap (3, 5]$



Have you completed the above graphs? Please do before going further. Check your work with the graphs on page 15.

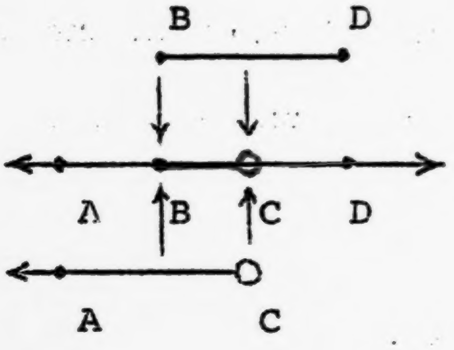
Are they O.K.? Do you feel comfortable with the task of constructing graphs of sets of points on a line? Why don't we look at the U or ∩ of some lines, rays, line segments, and half-lines and see if we can construct their graphs. The procedure we can use is the same as that we have been using with a number line. A few examples are found on the following two pages.

i. $\overrightarrow{BC} \cap \overrightarrow{CA}$



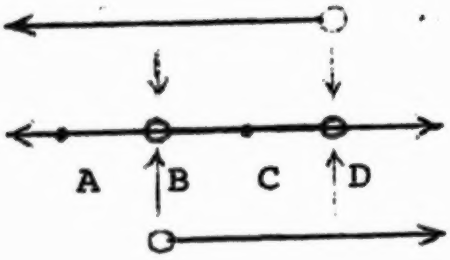
In (i.) the set of points $\overline{BC}$, which is common to the two rays, is shown in red. $\overrightarrow{BC} \cap \overrightarrow{CA}$ is found by projecting each ray onto the line and noting their $\cap$. Remember that $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are actually sets of points on the line but which are shown above and below the line $\overleftrightarrow{AD}$ for clarity.

ii. $\overline{BD} \cap$ half-line CA



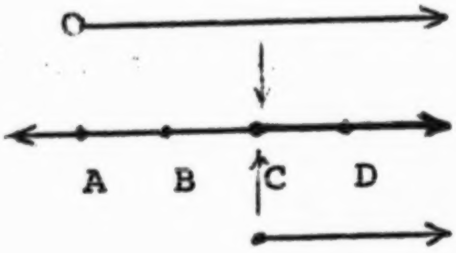
In (ii.) the set of points on $\overleftrightarrow{AD}$ which is the $\cap$ of $\overline{BD}$ and the half-line CA is shown in red. Note carefully that there is an open dot at C. Although C is a point on $\overline{BD}$, C is not a point on half-line CA. It is true that C is the end-point of half-line CA, but recall that end-points are not included as a part of half-lines.

iii. half-line DA $\cap$ half line BD



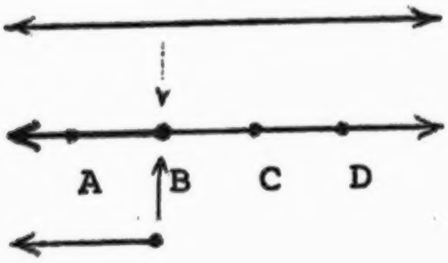
In (iii.) notice that the letters were left off the set above the line and the set below the line. This is usually done without loss of clarity because these are sets which are actually on the line $\overleftrightarrow{AD}$. The $\cap$ does not include the end-points of either half-line.

iv.



In (iv.) the intersection shown in red is $\overrightarrow{CD}$.

v.



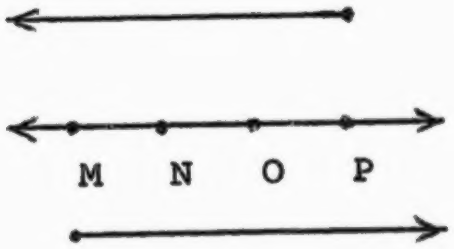
In (v.) it is shown that $\overleftrightarrow{AD} \cap \overrightarrow{BA} = \overline{BA}$. Notice that the entire line $\overleftrightarrow{AD}$ is shown above itself while $\overrightarrow{BA}$ is shown below the line $\overleftrightarrow{AD}$. Remember, this is only for clarity in picturing what is going on.

You are probably ready to try some on your own. Great! If you're with it, we can move on to greener pastures. Try your skill on the following problems.

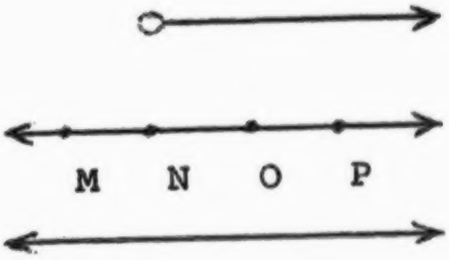
One set of points on the line $\overleftrightarrow{MP}$ is represented above $\overleftrightarrow{MP}$ while a second set of points on $\overleftrightarrow{MP}$ is represented below $\overleftrightarrow{MP}$.

IDENTIFY the $\cap$ of the two sets of points in each of the following by shading line $\overleftrightarrow{MP}$ in red over the appropriate portion.

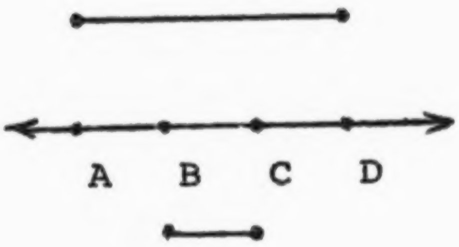
9.



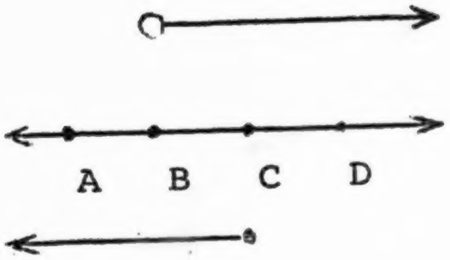
10.



11.



12.

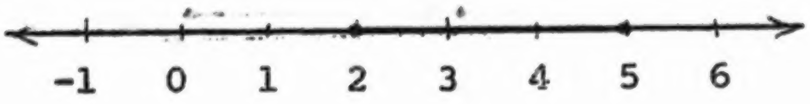


You'll find solutions to these problems on page 15 and 16

Answers to Performance Tasks

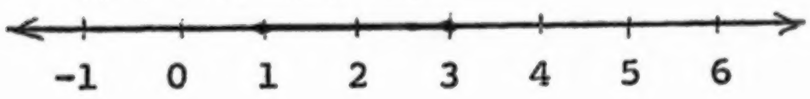
Page 4

1.

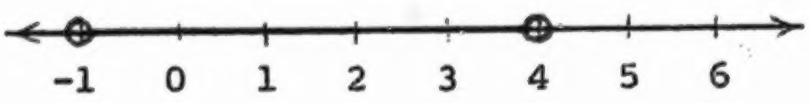


Page 7

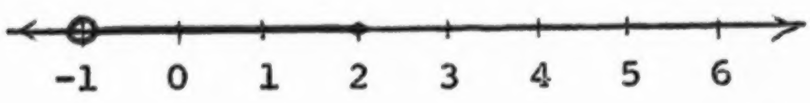
2. [1, 3]



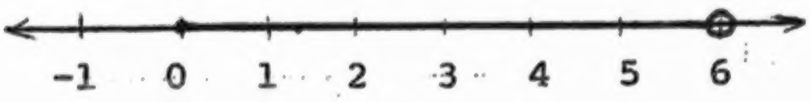
3. (-1, 4)



4. (-1, 2]

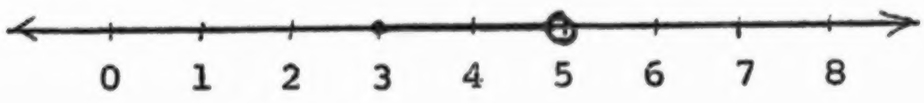


5. [0, 6)

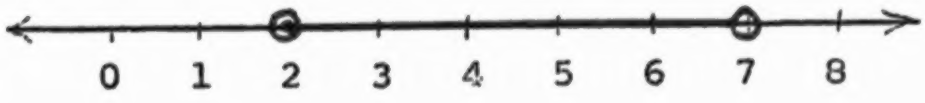


Page 10

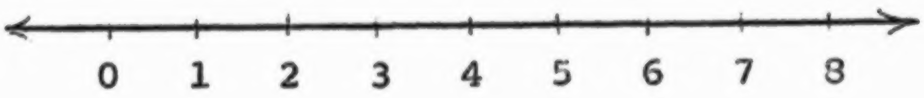
6. $[2, 5) \cap [3, 7]$



7. $(2, 5] \cup [3, 7)$

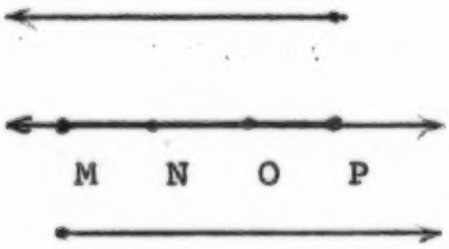


8. $[0, 3] \cap (3, 6]$

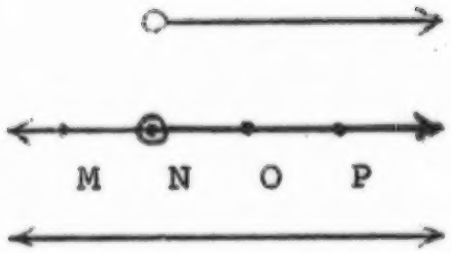


Page 13

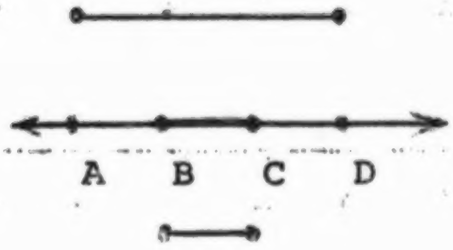
9.



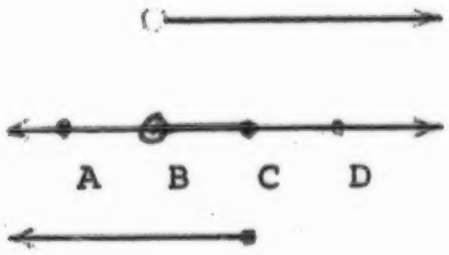
10.



11.



12.

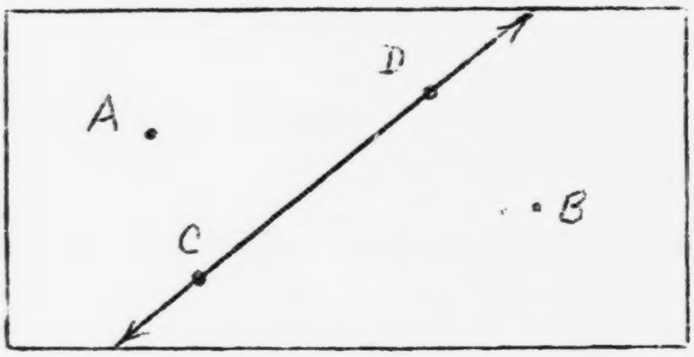


IV. The Plane

The sides of a box are representations of a special set of points called planes. We do not define a plane, but we can think of it as an ideal flat surface. Hence, any object which appears flat we call a representation of a __________. I hope you wrote the word "plane" in the blank. Otherwise, the sentence would not make very much sense, would it? Try the word "dog" in the blank and see what I mean.

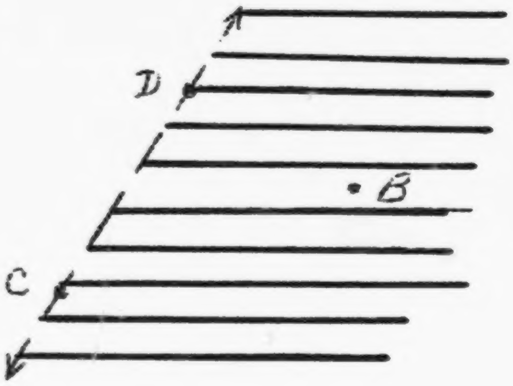
What are some other such representations of planes? Yes, a chalkboard would be one. What about a wall, a windowpane, a sheet of paper, or a floor? Like the line, it is important to note that the planes represented by the above objects actually extend on and on in space.

We can draw a line on a sheet of paper. Since the sheet of paper represents a set of points called a plane, it is not difficult for us to acknowledge that you can draw a line on a plane. Let's consider the rectangle on the following page to be a representation of a plane.



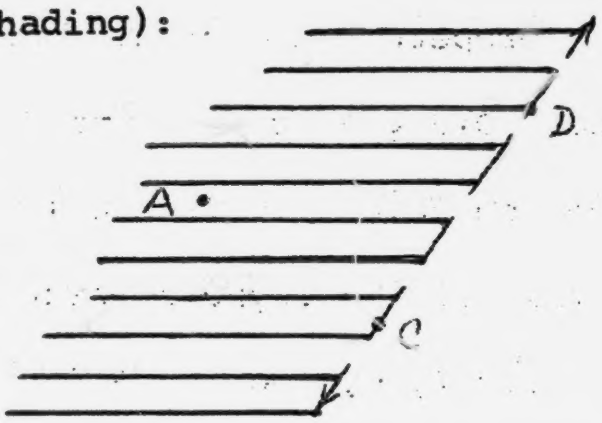
Any line $\overleftrightarrow{CD}$ of a plane separates the plane into two half-planes and the line $\overleftrightarrow{CD}$. The line $\overleftrightarrow{CD}$ is described as the common edge of the half-planes even though it is not a line of either half-plane. Let A be a point of one half-plane and let B be a point of the other half-plane, then the half-planes may be called A-CD and B-CD.

Suppose we wished to illustrate just half-plane B-CD. Since $\overleftrightarrow{CD}$ is not a part of the half-plane, we will dash $\overleftrightarrow{CD}$ to represent its location but at the same time excluding all points on $\overleftrightarrow{CD}$ as part of half-plane B-CD. We'll shade the half-plane in blue to be more specific about which half-plane we mean:



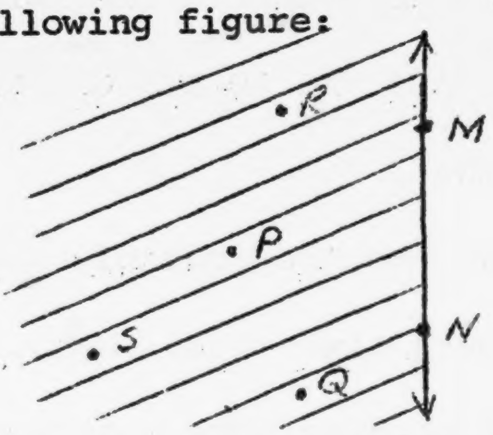
The shading extends on and on on the side of line CD that point B is located on.

Half-plane A-CD would be represented as follows (red shading):



You have probably noticed that if you have three points, two of which are on a line while the third point is not on the line, then you have a line and a half-plane represented.

Let's look at this a little more carefully. Suppose we have the following figure:



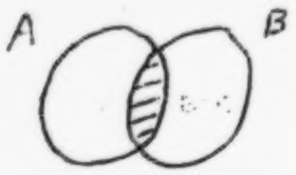
The point P is a point on the half-plane P-MN. The line $\overleftrightarrow{MN}$ is shown solid, therefore the $\cup$ of the half-plane P-MN and the line $\overleftrightarrow{MN}$ is represented in the figure. The point Q is on the same half-plane that P is on. Thus, we could call the half-plane Q-MN. That is, any point on the half-plane can be used to name the half-plane.

Point R is also on the half-plane. That is, the points P, Q, and R have a property in common: they are in the __________ half-plane. Did you write "same"? That is correct, they are each on the same side of the line $\overleftrightarrow{MN}$.

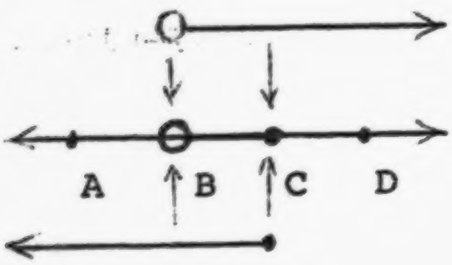
Let me re-phrase this idea. Suppose point R is on the same half-plane as points P, Q, and S. Before going further, let's note that any one of these points can be thought of as being used to specify which half-plane is being referred to. Now, we have given that R is on the same half-plane as P, Q, and S. O.K., then if R is to the left of line $\overleftrightarrow{MN}$, it follows that P, Q, and S are to the left of line MN. In fact, IF ONE POINT ON THE HALF-PLANE IS TO THE LEFT OF LINE MN, THEN ALL THE POINTS OF THE HALF-PLANE ARE TO THE LEFT OF LINE MN.

Sounds foolish? You're probably saying it's too obvious to spend so much time with. Well, we'll leave it for now and look at something more interesting.

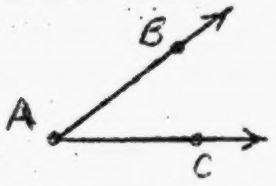
We've looked at the intersection of Venn diagrams:



and of rays, line segments, etc.:



What would you have if you had the union of two rays which do not lie on the same number line? Let's look at such a U .



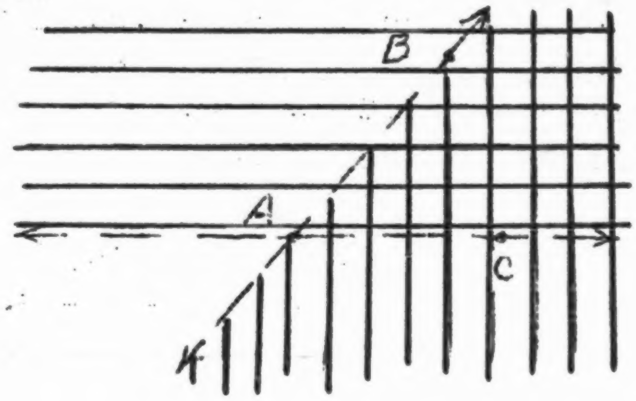
Notice that the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have a common end-point A.

Definition 8: Any figure formed by two rays that have a common end-point is a plane angle.

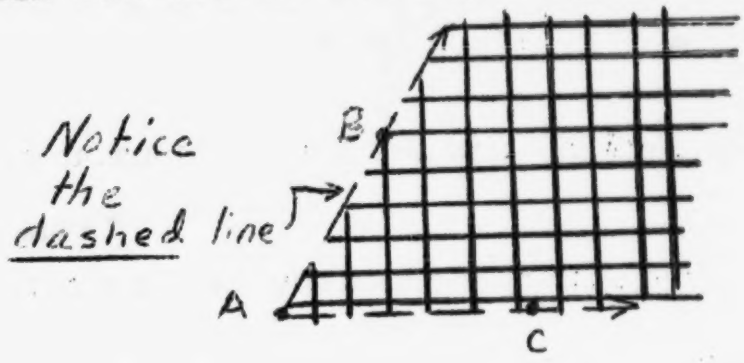
The angle in the figure may be designated as $\angle$ BAC. That is, $\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$. Note that the common end-point A, referred to as the vertex of the angle, is located in the middle position of the designation $\angle$ BAC.

The rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called sides of $\angle$ BAC. The angle consists only of the rays, it does not include the region in between the rays. The region in between the rays, is named the interior of $\angle$ BAC.

If the half-plane (shaded in red) that contains B and has edge $\overleftrightarrow{AC}$ intersects the half-plane (shaded in blue) that contains C and has edge $\overleftrightarrow{AB}$, their intersection is called the interior of $\angle$ BAC.



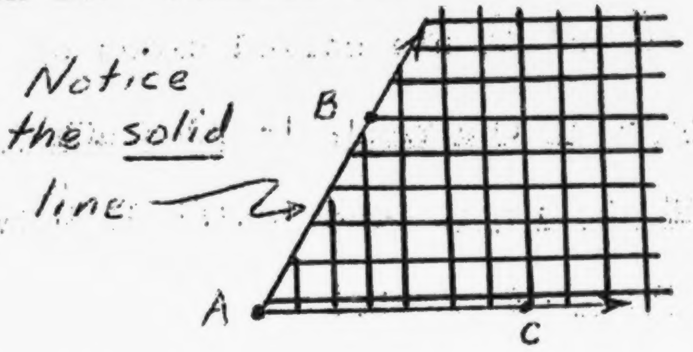
The double shading identifies the interior of $\angle BAC$. Removing the excess shading and deleting the parts of lines $\overleftrightarrow{AB}$ and $\overrightarrow{AC}$ which are not $\overrightarrow{AB}$ and $\overrightarrow{AC}$, respectively, we have



The name of the $\cap$ set would be:

(Interior of $\angle BAC$).

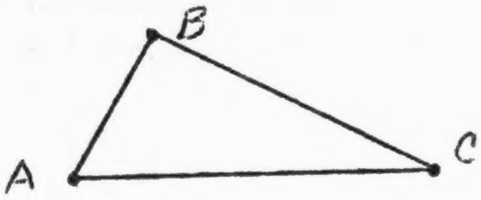
However, suppose the $\angle BAC$ was included in the figure with its interior as:



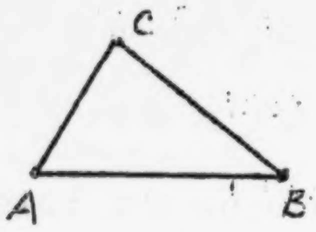
The name of the $\cap$ set would then be

$\angle BAC \cup$ (Interior of $\angle BAC$).

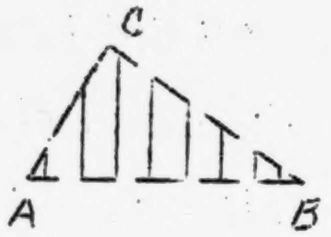
A triangle has three sides which are three line segments. Thus, triangle ABC (i.e., $\triangle ABC$) consists of the points of $\overline{AB} \cup \overline{BC} \cup \overline{CA}$.



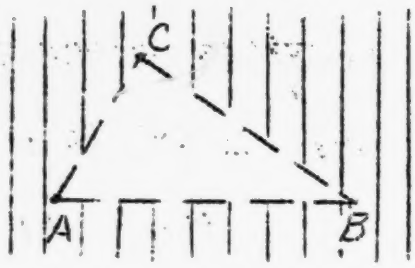
The interior of △ ABC consists of the points in the intersection of the interiors of the three angles, ∠ ABC, ∠ BCA, and ∠ CAB. The exterior of △ ABC consists of the points of the plane that are neither points of the △ nor points of the interior of the △ .



△ABC

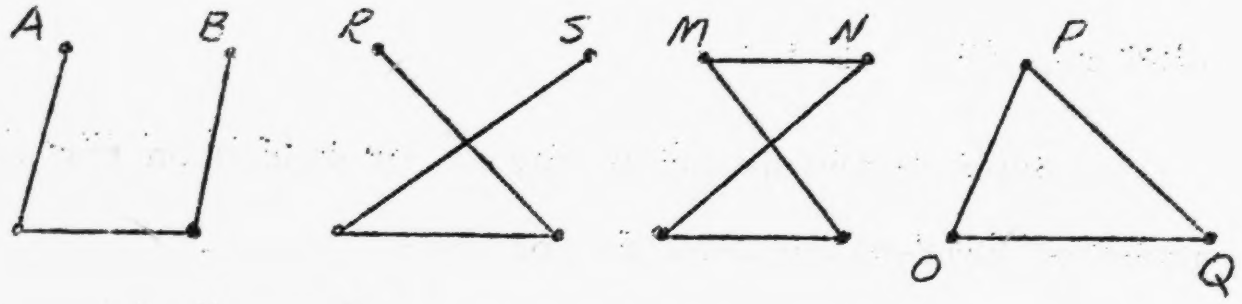


Interior of △ABC



Exterior of △ABC

A triangle is one form of a simple closed figure. But what is a __________ __________ __________? A broken line is any connected union of line segments.



The diagram shows four broken lines. Broken lines RS and MN cross themselves while broken lines AB and OPQ do not.

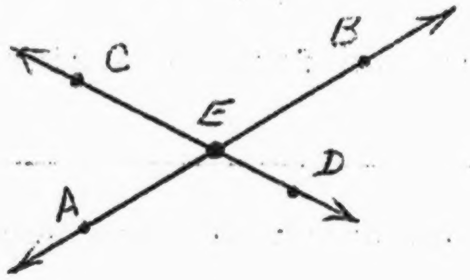
Definition 9: A figure (such as broken lines AB and OPQ) which does not cross itself is a simple figure.

The figures MN and OPQ differ from figures AB and RS. Each of the figures MN and OPQ starts at a point and returns to that point. We say that such a figure is closed.

Hence, $\triangle$ OPQ is a __________ __________ __________.

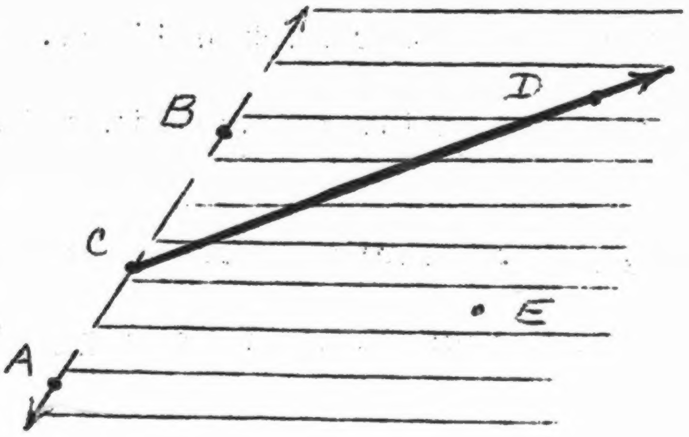
Stop! Fill in the blanks! If you can not, reread the two previous paragraphs. You did fill them in? By now you know why $\triangle$ OPQ is a simple closed figure. Let's move on.

The $\cap$ of two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ would be the point E shown in red as

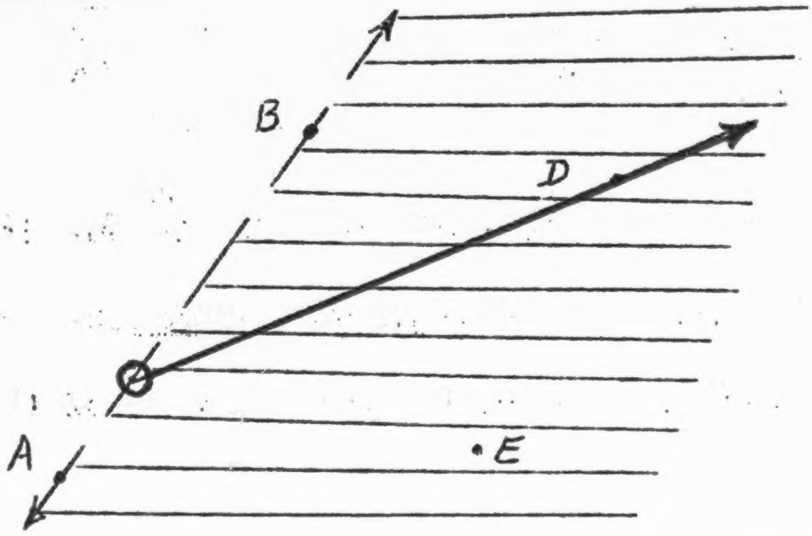


That is, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = E$

Suppose the graph of ray $\overrightarrow{CD}$ is placed on top of the graph of half-plane E-AB as follows:



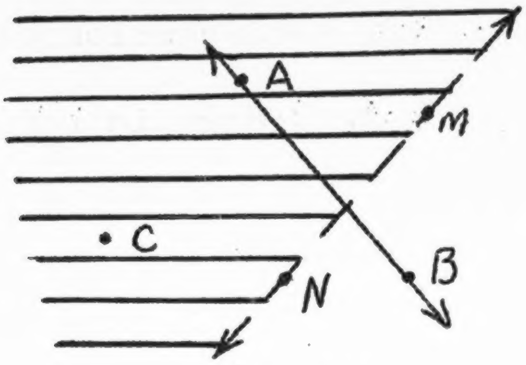
What is the ∩ of the two sets of points? The ∩ of the two sets of points are those points which are common to both sets. The dashed line $\overleftrightarrow{AB}$ shows the location of $\overleftrightarrow{AB}$ but denotes that the points of $\overleftrightarrow{AB}$ are not included in the figure. We recall that half-plane E-AB does not include the points on line _____ . Did you write $\overleftrightarrow{AB}$? You are correct! Although point C is in $\overrightarrow{CD}$, point C is not in half-plane E-AB. That is, point C is not common to both sets of points. Hence, point C is not included in the intersection set. By looking at the above figure we can see all the points of $\overrightarrow{CD}$, except C, are common to half-plane E-AB. Therefore, the intersection of half-plane E-AB and $\overrightarrow{CD}$ is half-line E-AB (shown in red in the figure below).



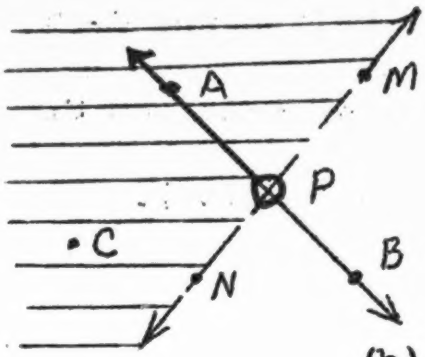
We have, (half-plane E-AB) ∩ $\overrightarrow{CD}$ = ______________ .

You are correct if you named the $\cap$ set of half-plane E-AB and $\overrightarrow{CD}$ to be half-line CD.

Let's look at some other sets of points and try to identify and name their intersection sets. We will be looking for those points which are common to both sets. The green graph of half-plane C-MN is shown in (a) below with the graph of line $\overleftrightarrow{AB}$ shown in black on the same plane. Their intersection (those points which are common to both) is identified in red in (b). The name of their intersection set is half-line PA.



(a)

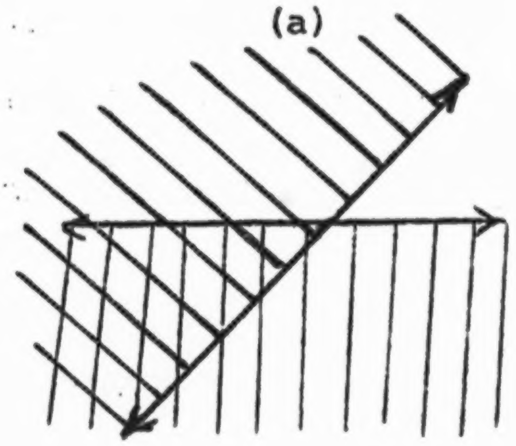


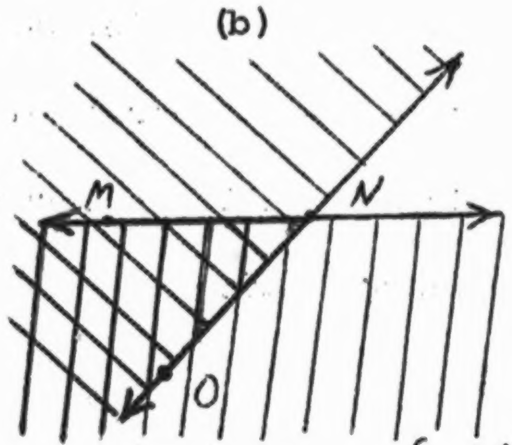
(b)

Notice that point P is part of line $\overleftrightarrow{AB}$ but is not part of half-plane C-MN. Since P is not common to both the line $\overleftrightarrow{AB}$ and half-plane C-MN, then P is not part of their intersection set.

In each of the following figures, two graphs are shown plotted on the same plane. Under (a) one graph is shown in green and a second graph is shown in black. Under (b) the intersection of the two graphs is identified in red; the name of the intersection is also given. Remember that the intersection of any two graphs consists of the points which are common to both graphs. THE INTERSECTION SET IS NAMED BY ASSIGNING LETTER NAMES TO POINTS AND USING THESE LETTERS TO NAME THE INTERSECTION SET.

i.

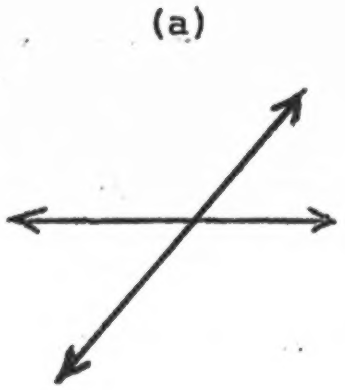


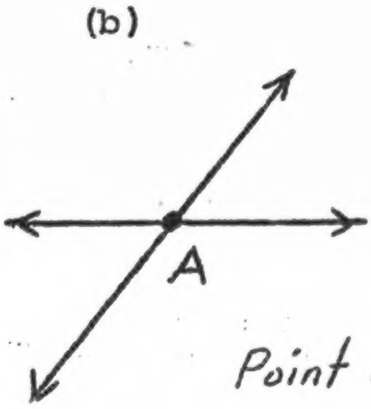


∠MNO ∪ (Interior of ∠MNO)

In (i.) the ∠ MNO is common to both sets. In addition, the interior of ∠ MNO is common to both sets. Hence, the intersection consists of the ∠ MNO in union with its interior. Notice that the points which are common can be located by noting where both green and black overlap in (a).

ii.



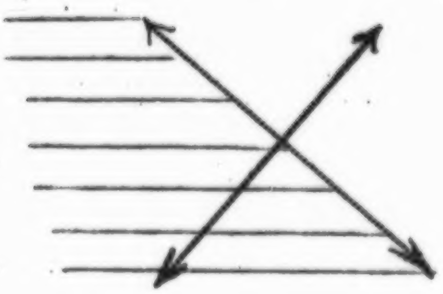


Point A

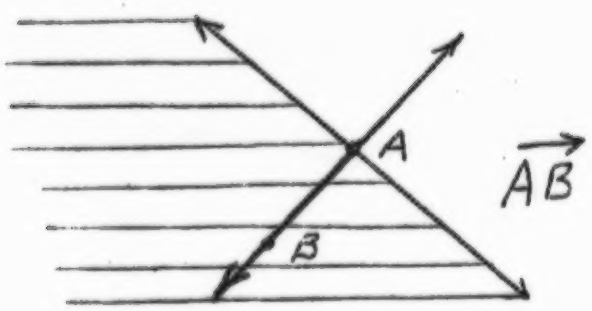
In (ii.) the situation is similar to the figure on page 8.

iii.

(a)



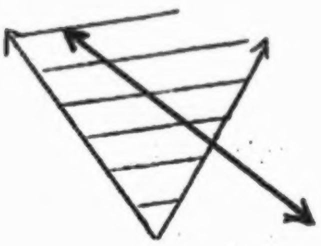
(b)


A
B
$\overrightarrow{AB}$

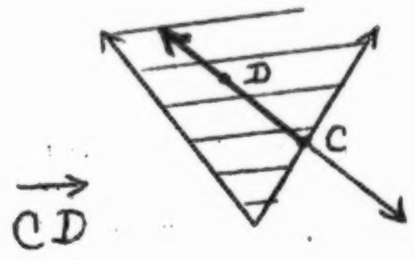
In (iii) the line shown in black under (a) is solid, therefore point A is common to both sets. Compare this intersection with the one shown on page 10. The line $\overleftrightarrow{MN}$ was dashed on page 10.

iv.

(a)

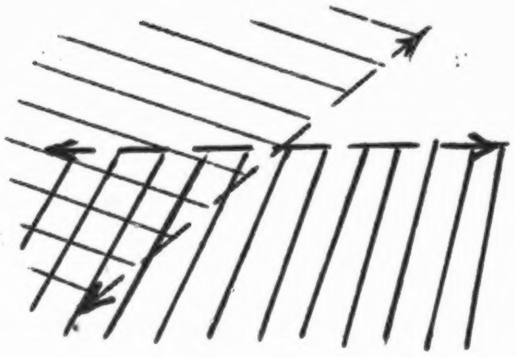


(b)

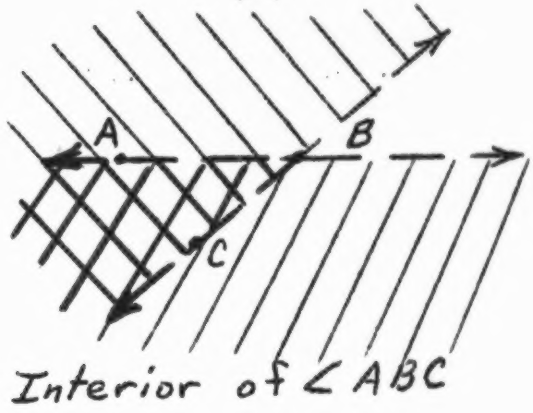

D
C
$\overrightarrow{CD}$

v.

(a)



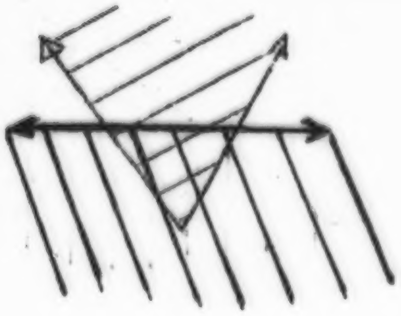
(b)


A
B
C
Interior of ∠ABC

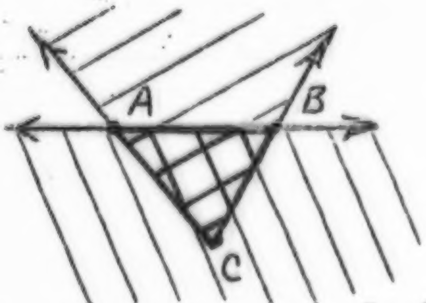
In (v.), since the sides of ∠ ABC are dashed, all points on ∠ ABC are excluded. However, the interior of ∠ ABC is common to both sets in (a). Hence, the intersection set is the interior of ∠ ABC.

vi.

(a)



(b)

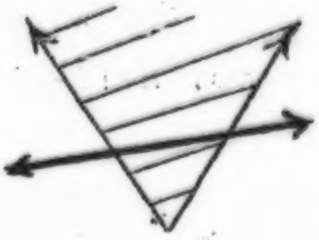


$\triangle ABC \cup$ (Interior of $\triangle ABC$

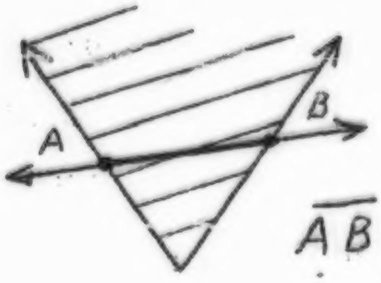
In (vi.) the sides of $\triangle$ ABC are solid and are common to both sets in (a). Also, the interior of $\triangle$ ABC is common to both sets.

vii.

(a)



(b)



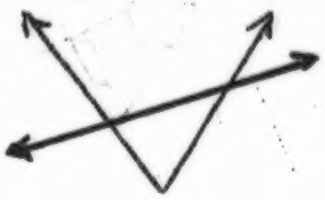
$\overline{AB}$

In (vii.) the line segment $\overline{AB}$ is common to both sets.

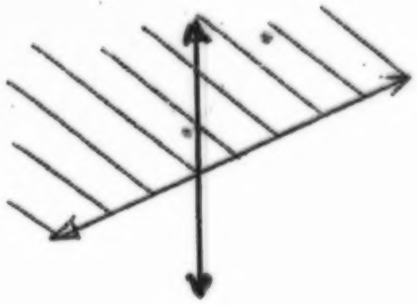
Practice identifying intersection sets by performing the following tasks:

IDENTIFY (by coloring in red) the $\cap$ of the two graphs in each of the following cases:

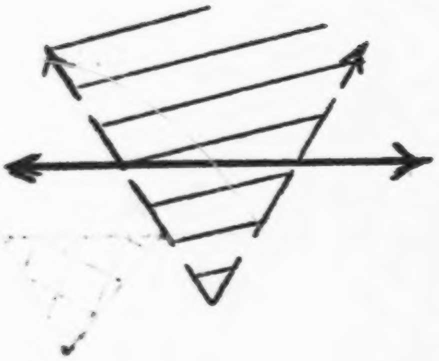
1.



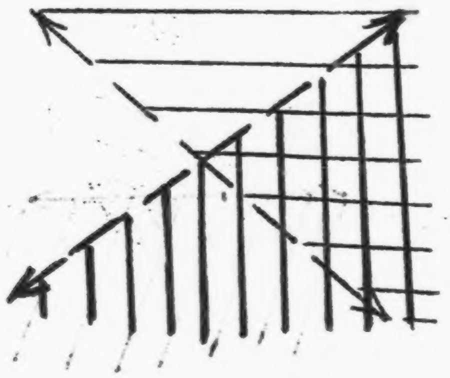
2.



3.



4.



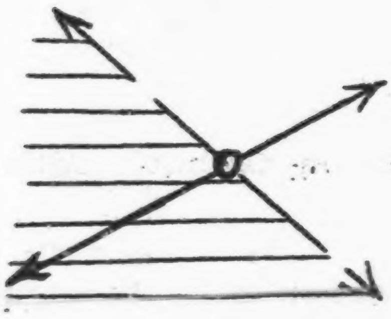
Solutions to the four problems are shown on page 15.

Now, you can practice naming intersection sets by performing these tasks.

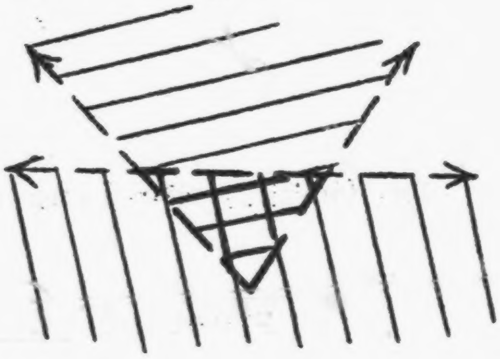
GIVEN the $\cap$ set (shown in red) of two graphs in each of the following cases,

NAME the $\cap$ set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

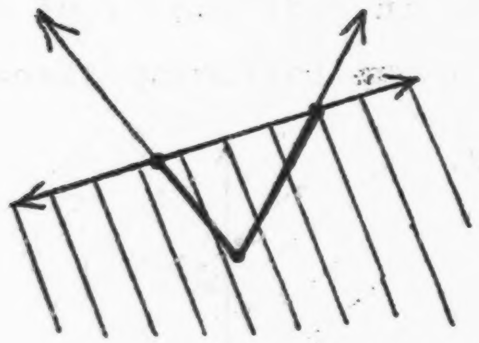
5.



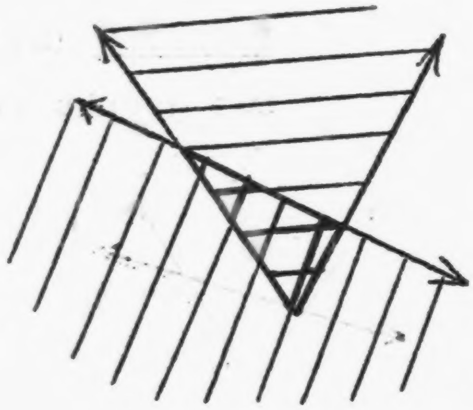
6.



7.



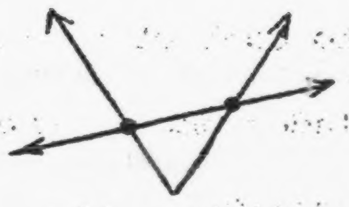
8.



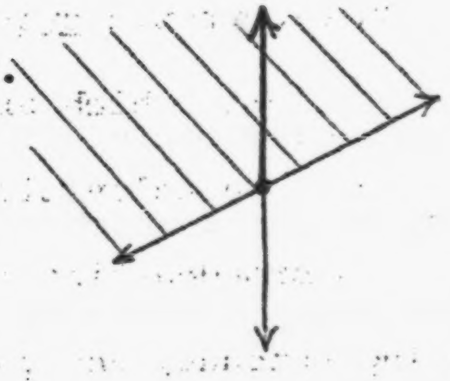
Names for each of the above are given on page 15.

Answers to Performance Tasks

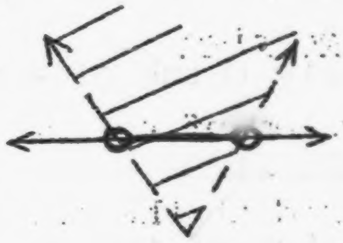
1.



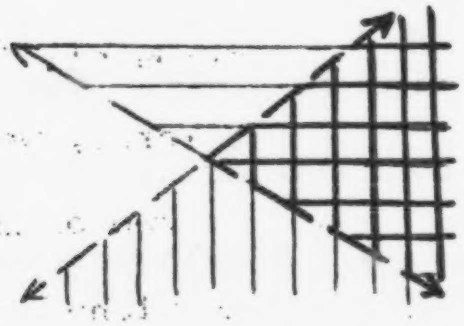
2.



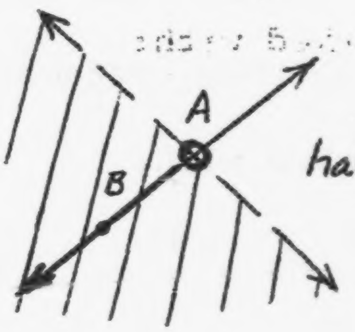
3.



4.

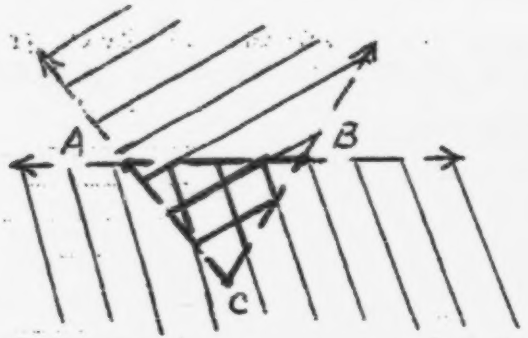


5.



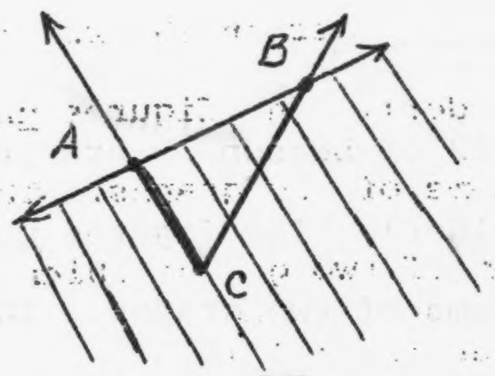
half-line AB

6.



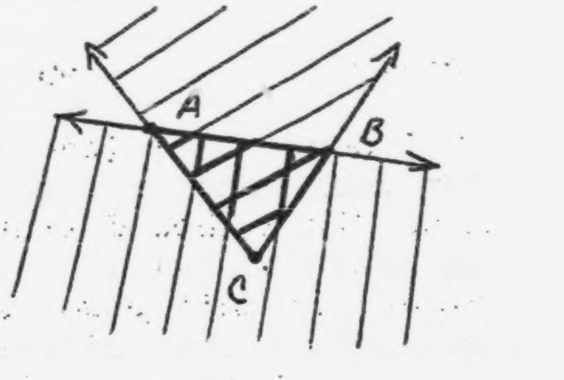
Interior of $\triangle ABC$

7.



$\overline{AC} \cup \overline{CB}$

8.

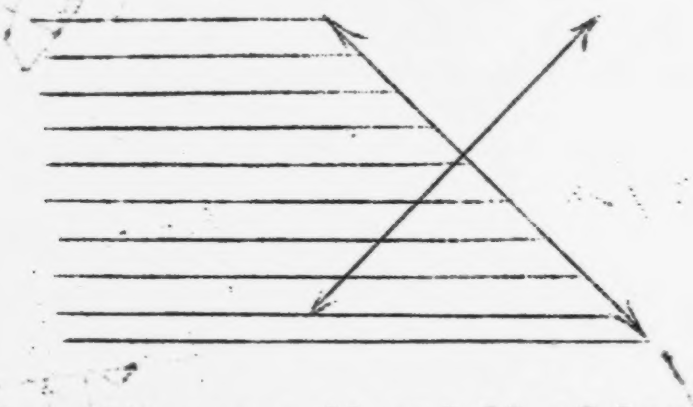


$\triangle ABC \cup$ (Interior of $\triangle ABC$)

V. More on the Plane

An electronic system usually refers to a group of electrical parts which are connected to each other in a particular way. A television system includes among other things: a broadcasting station, a transmitting antenna, a receiving antenna, and your TV set. The key point in referring to such a system is that we normally think of a system by considering all the parts at the same time.

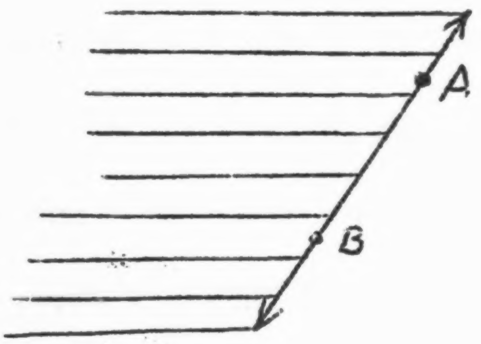
Similarly, we can think of a system of two or more graphs as: two or more graphs considered at the same time on the same plane. In Lesson IV we worked with systems consisting of two graphs without calling them systems. Here is one system of two graphs that we worked with:



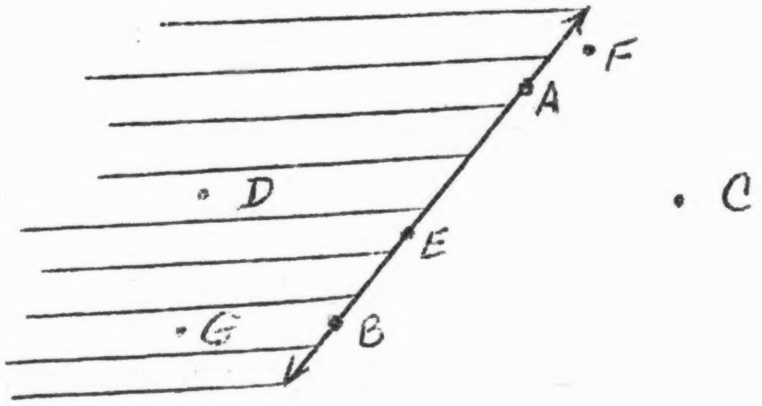
For your benefit pages 11 and 12 of Lesson IV are provided in Appendix A-1 (the yellow folder). The figures under (a) in Appendix A show different systems of two graphs. The above figure on this page is a system of two graphs which is shown on page 12 in Appendix A-1 under (iii)-(a).

Now notice that on pages 11 and 12 of Appendix A the intersection of the *system of two graphs* is shown in red *under (b)* for each of the cases: (i), (ii), (iii), (iv), and (v). Recall that an *intersection* of two graphs consists of the *points which are common to both graphs*. Another way of saying this is that an *intersection* of two graphs consists of the *points which satisfy the conditions of both graphs*.

What do we mean by the ____________________ of both graphs? Let's look first at what we mean by the conditions of one graph. Consider the graph:



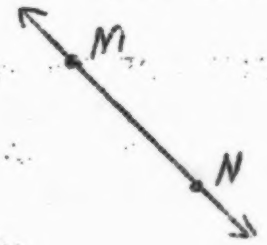
For a point on the plane *to satisfy the conditions of this graph* it must be located either on line $\overleftrightarrow{AB}$ or in the shaded region to the left of line $\overleftrightarrow{AB}$. For example, let's name some points on the plane and see whether or not they satisfy the conditions of the above graph. Consider the points C, D, E, F, and G shown on the next page.



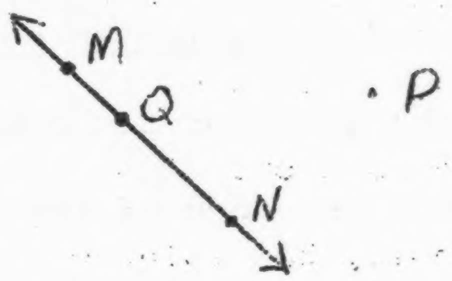
Point C does not lie on $\overleftrightarrow{AB}$ or in the shaded region to the left of $\overleftrightarrow{AB}$ (that is, C does not lie in half-plane D-AB). Therefore, point C does not satisfy the conditions of the graph. What about point D? Does D satisfy the conditions of the graph? Yes, it does! It lies in the shaded region of the graph and thus satisfies the condition of the graph that its points must lie either in the shaded region or on the line $\overleftrightarrow{AB}$. Note, that point E is on line $\overleftrightarrow{AB}$. Hence, E also satisfies the conditions of the graph.

Now, let's give you a chance to express yourself. Point F ____________________ the conditions of the graph. Did you write "does not satisfy"? You are correct, because F is neither on line $\overleftrightarrow{AB}$ or in the shaded region. Point G ____________________ the conditions of the graph. Yes, you're correct if you wrote "satisfies" because point G is in the shaded region. Point A ____________________ the conditions of the graph. Point A lies on the line $\overleftrightarrow{AB}$, therefore point A "satisfies" the conditions of the graph.

Let's look at another graph:



For a point on the plane to satisfy the conditions of this graph, the point must lie on the line. Now, let's consider other points on the plane:



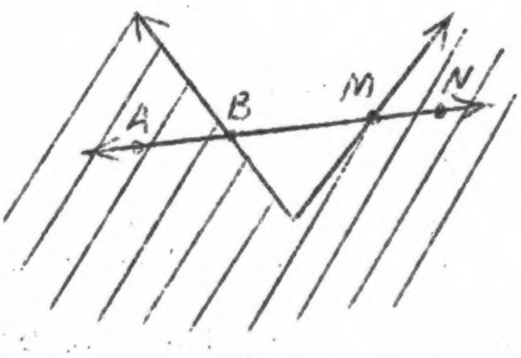
P does not lie on $\overleftrightarrow{MN}$ and therefore does not satisfy the conditions of the graph. However, since point Q does lie on the line $\overleftrightarrow{MN}$, Q ____________________ the conditions of the graph. Correct! Q "satisfies" the conditions of $\overleftrightarrow{MN}$.

For a point on a plane to satisfy the conditions of a system of two graphs, it must satisfy the conditions of both graphs. That is, a point must be common to both graphs if it is to satisfy the conditions of both graphs. But points which are common to both graphs are in the ____________________ set of the two graphs. Did you write "intersection"? You're with the idea! We recall that the definition of an intersection set of two graphs is that set whose points are common to the two original graphs.

What does this all lead to? We can now say with clarity that POINTS WHICH ARE IN THE INTERSECTION SET SATISFY THE CONDITIONS OF BOTH GRAPHS. That is, the intersection set satisfies the conditions of both graphs.

Given a system of two or more graphs, the set of points that is common to both graphs (i.e., the intersection set) may be called the solution set of the system. Each of the points in the solution set satisfies the conditions of the system. That is, each of the points in the solution set satisfies the conditions of both graphs.

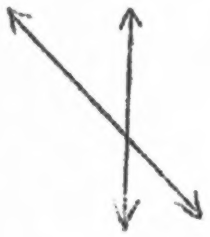
For example, the solution set of the following system of two graphs



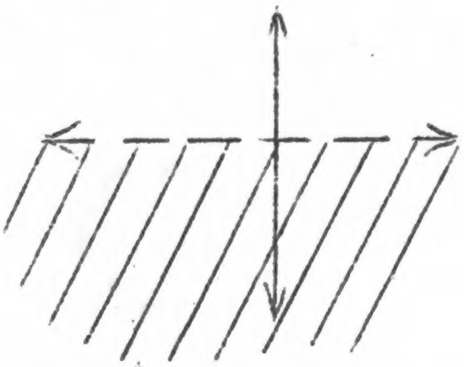
is $(\overrightarrow{BA} \cup \overrightarrow{MN})$. The solution set of the system is precisely the intersection set of the two graphs in the system.

In each of the following cases, a figure of a system of two graphs is given. Name the solution set of each system:

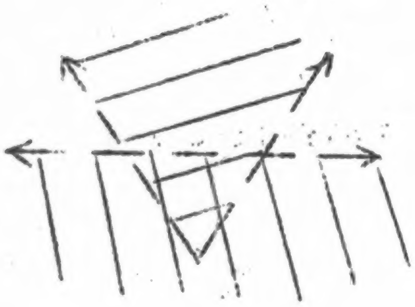
1.



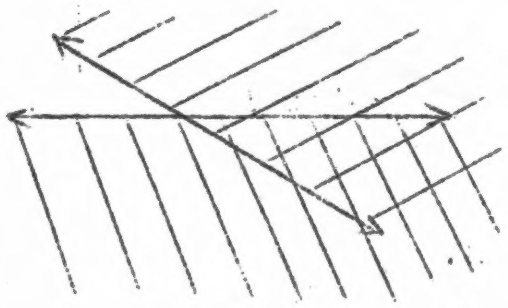
2.



3.



4.

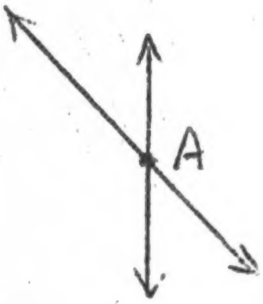


The solutions to each of the four problems are given on page 7 .

Answers to the Performance Tasks

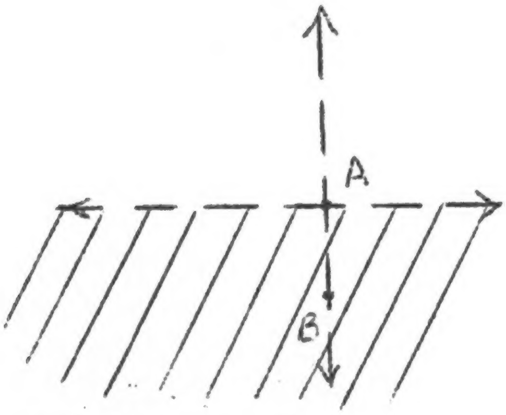
Page 6 .

1.



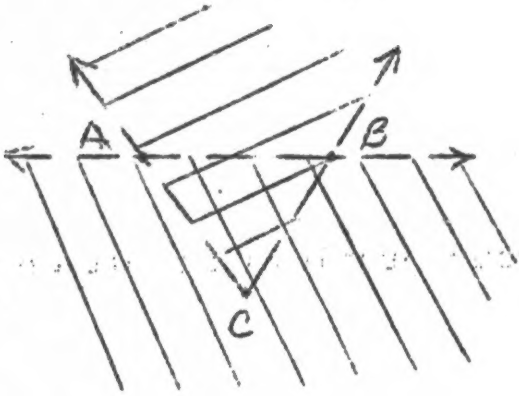
Point A

2.



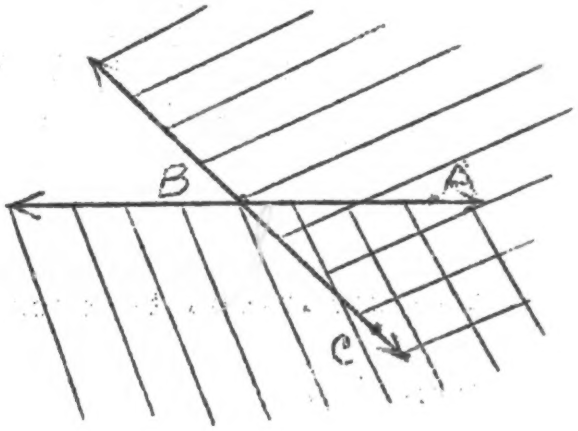
half-line AB

3.



Interior △ ABC

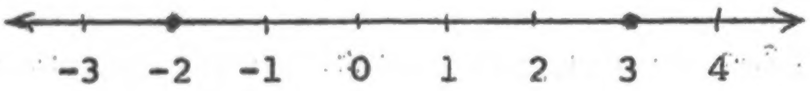
4.



∠ ABC ∪ (Interior of ∠ ABC)

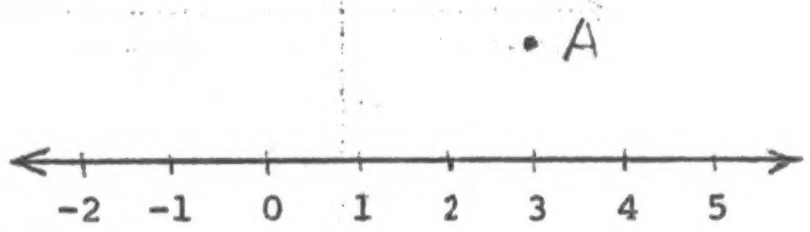
VI. Ordered Pairs

We have worked with the number line and have practiced locating points on the line that correspond to specific integers. The two heavy dots on the following number line are the points on the number line which correspond to -2 and 3, respectively, reading left to right on the line.

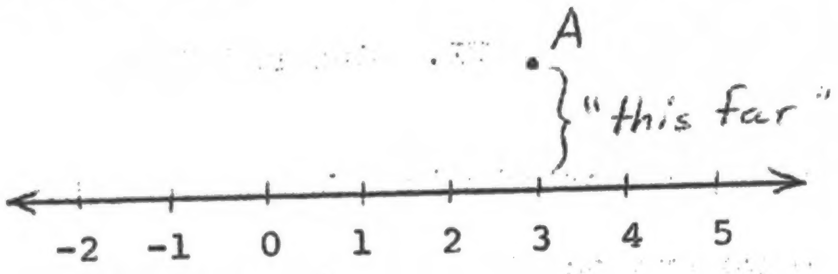


Locating points on a number line is no problem, but how do you locate a point on a plane?

Perhaps another question should be first answered: Given a point on a plane, how do you name the point so that the name tells the point's location? Consider the following situation:

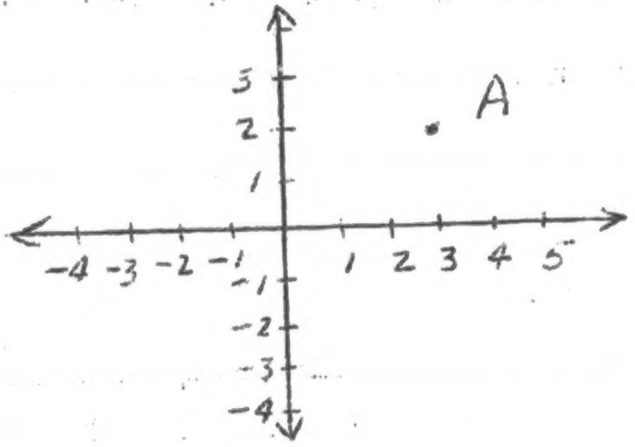


How do you tell someone where A is located on the plane? We can easily say you go to the right of 0 three units to the point on the number line corresponding to the integer 3. Then---well, then we go up above the number line some distance, about "this far." By "this far" we mean the vertical distance between the number line and the point A.



It would be very handy if we could place a ruler and measure the vertical distance. But there may be other points for which we'll want to measure the vertical distance from the number line. So let's just place a permanent ruler (which is in effect a vertical number line) perpendicular to the horizontal number line. Let's permit the 0 point on the vertical number line and the 0 point on the horizontal number line to overlap.

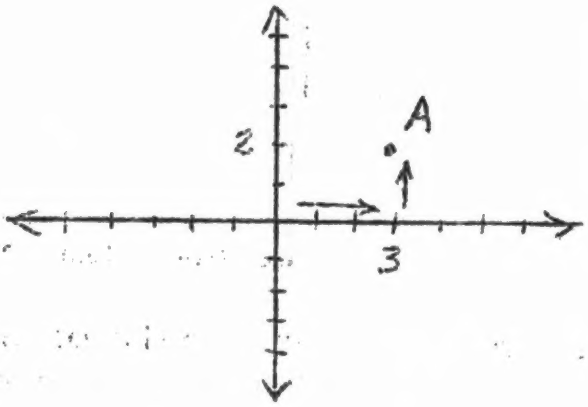
We have



For convenience, let's name the horizontal number line the x axis and name the vertical number line the y axis. When two axes are specified on a plane in this manner, we call the plane a Cartesian plane.

We can locate point A by saying:

> Starting from 0, go out 3 units
> on the x axis, then go up 2 units
> parallel to the y axis.



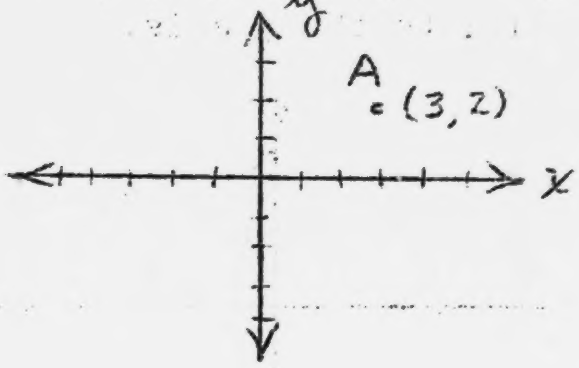
That's a lot of words for describing one point. Let's try simplifying the description. How about:

3 out on x
and 2 up on y ?

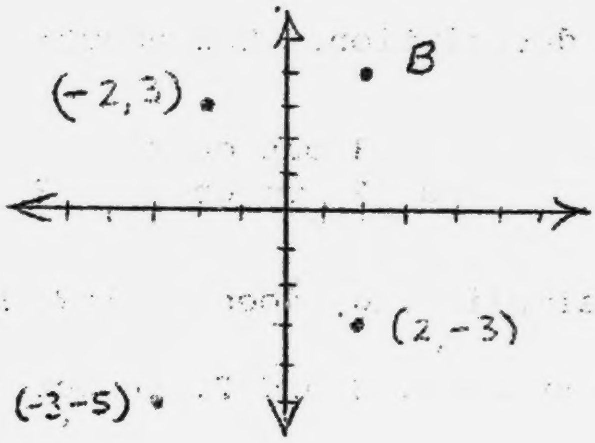
It still needs simplifying, doesn't it? Notice that we're using a pair of numbers, 3 and 2, in our efforts to describe A's location. Also, notice that we've been using the distance along the x axis first each time. That is, we've been using the pair of numbers in a certain order--the x distance first and the y distance second.

For convenience we could symbolize such an ordered pair of numbers by the symbol (3, 2) . By placing the 3 first we mean by this symbol: 3 units from 0 along the x axis. By placing the 2 second we mean: 2 units from 0 along the y axis.

Therefore, the location of point A can now be described as (3, 2) and denoted on the plane by



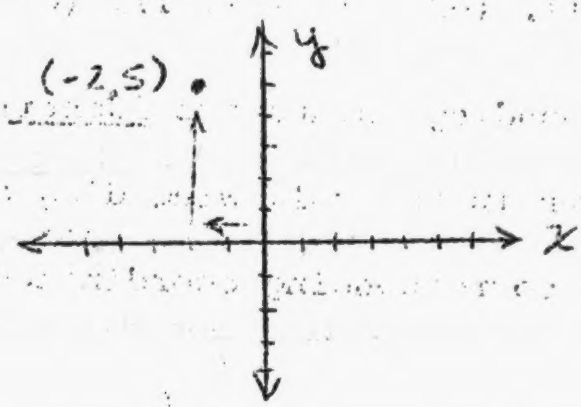
Similarly we can locate and name other points on a Cartesian plane. These ordered pair of numbers are called



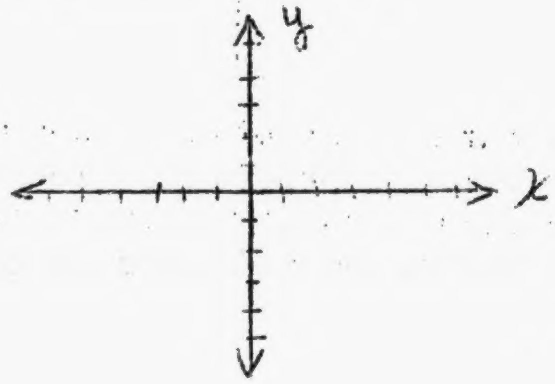
the coordinates of the points they locate in the plane. What are the coordinates of point B in the figure above? ________. You are correct if you wrote (2, 4) .

In each of the cases the distance along the x axis was noted first, followed by the distance along the y axis. From now on, we can use the symbol (x, y) for an ordered pair. If we were given the value of x and of y we would be able to locate the corresponding point on the Cartesian plane. Suppose x = -2 and y = 5 for the ordered pair (x, y) . We would then have the location of a specific point whose coordinates are (-2, 5).

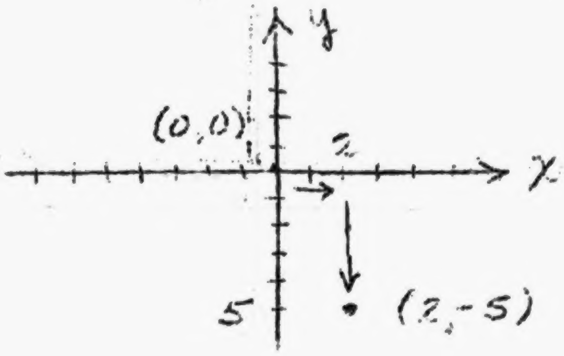
In order to locate point (-2, 5) on the Cartesian plane we move from 0 on the x axis to the left 2 units and then move vertically 5 units up parallel to the y axis.



I'm sure you are ready to plot your own points on the Cartesian plane. Try plotting the point (2, -5) on the following Cartesian plane:



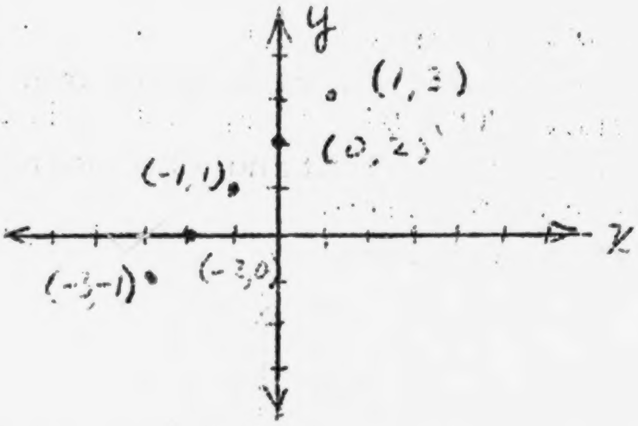
Here is how you should have plotted the point:



Every point on the Cartesian plane can be described by its corresponding ordered pair. The point (0, 0) shown on the above plane as the intersection of the two axes is called the origin. The braces { and } are used to denote a set of ordered pairs as follows:

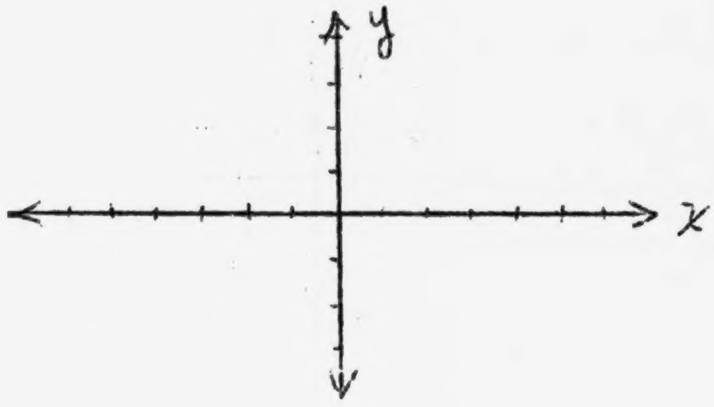
{(-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3)}

How the corresponding points are identified (or plotted) on a plane is shown in the following diagram. Note that each of the point's corresponding coordinates are given.

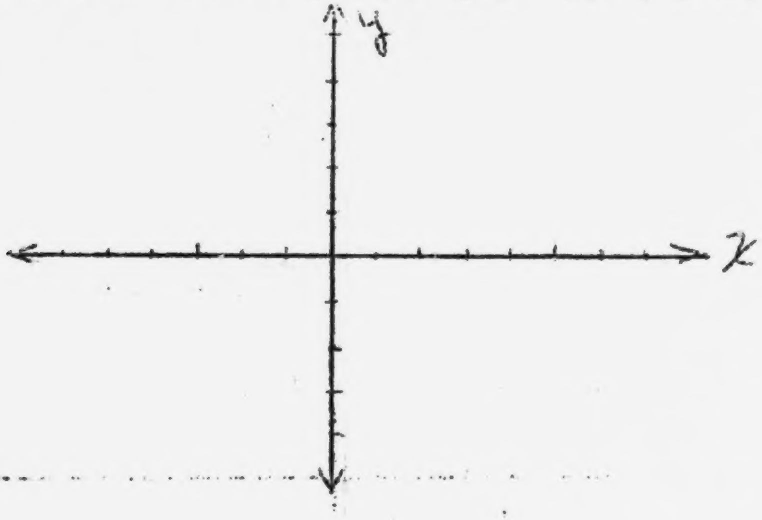


Now, it's your turn to practice. Identify the corresponding points for the following two sets of ordered pairs on the planes provided.

1. {(-6, 4), (-4, 2), (-2, 0), (0, 2), (2, 4)}



2. $\{(-2, 5), (-1, 4), (0, 3), (1, 2), (2, 1)\}$

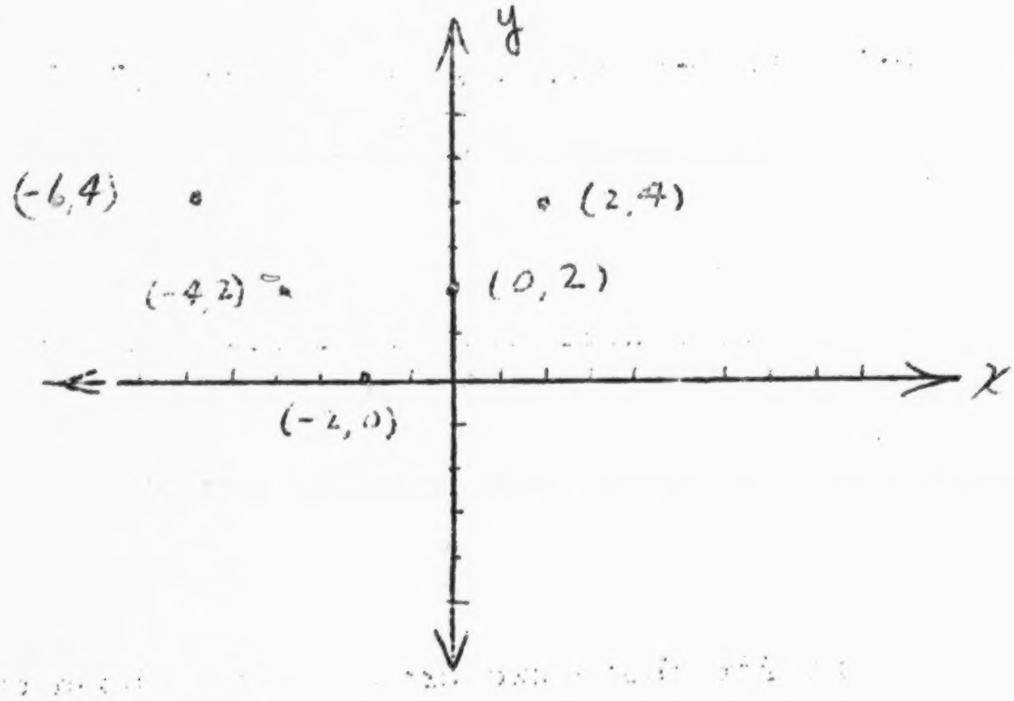


The solutions for these two problems are shown on page 8 .

Solutions to Performance Tasks

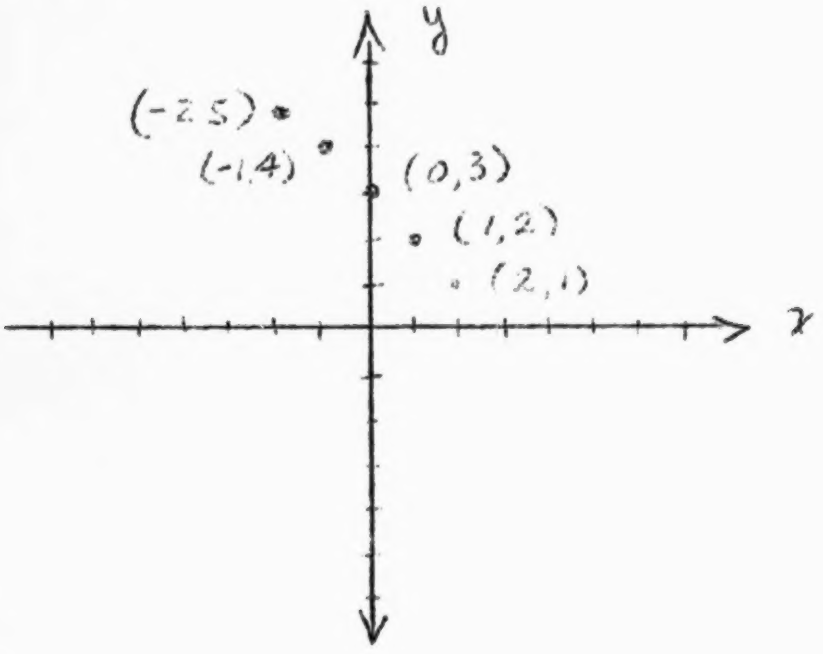
Page 6.

1.



Page 7.

2.



VII. Relations

Recall that in Lesson VI we plotted ordered pairs (x, y) where x is a coordinate on the x axis and y is a corresponding coordinate on the y axis. The ordered pair (-3, -1) denotes that the x coordinate and the corresponding y coordinate have a particular relationship: when x is -3, then y is -1. Similarly, the ordered pair (-2, 0) denotes a specific relationship between the x coordinate and the corresponding y coordinate: when x is -2, the corresponding y coordinate is 0.

Having noted that an ordered pair describes a particular relationship between the x coordinate and the corresponding y coordinate, let's clarify what is meant by an open number sentence.

Suppose you were told that

$$5 + 7 = 12$$

and that

$$9 - 3 = 4$$

Both of these expressions are called number sentences. Is it true that 5 + 7 = 12? Yes, of course. Is it true that 9 - 3 = 4? No, it is false. In both cases we were able to say whether the number sentence was true or false. When we can determine from the information available whether a number sentence is true or false, then the number sentence is called a statement. A statement can be a true statement or it may be a false statement:

5 + 7 = 12 is a true statement
9 - 3 = 4 is a false statement

There are some sentences for which the truth cannot be established with the information provided in the sentence:

__________ is the governor of Louisiana.

Such a sentence is called an open sentence. The "__________" is called a variable. You can vary what you write in the blank.

If you were given a set of names from which you could choose a name to place in the blank, the set of names would be called the domain of the variable.

Suppose you were given the following domain of the variable "__________":

{ Richard Nixon, W. C. Kennon, George Wallace } .

You now can choose any one of these three names to place in the blank of the open sentence:

__________ is the governor of Louisiana.

Of course, only one of the names satisfies the conditions of the sentence. The name Richard Nixon obviously does not satisfy the conditions of the sentence. Neither does the name George Wallace satisfy the conditions of the sentence. However, W. C. Kennon does satisfy the conditions. That is, W. C. Kennon substituted in the blank makes the sentence true, because the governor of Louisiana is W. C. Kennon.

Instead of the blank as the variable, we could have used an x and written:

x is the governor of Louisiana.
Domain of x is $\{$Nixon, Kennon, Wallace$\}$.

Similarly, we can write number sentences with a variable m:

$m + 3 = 7$, Domain of m is the set of integers

Since the number 4 is in the set of integers, we can select 4 from the domain of m to substitute for m . 4 satisfies the conditions of the number sentence $m + 3 = 7$. We say that the solution set of $m + 3 = 7$, where the domain of m is the set of integers, is the set of integers m such that $m + 3 = 7$. The solution set of integers m is $\{4\}$. That is, the number 4 makes the number sentence true; 4 solves the number sentence. Hence, $\{4\}$ is the solution set. $\{4\}$ is the set of integers m such that $m + 3 = 7$.

All of this can be written in set builder notation:

$$\{ m \mid m + 3 = 7 \text{ , m an integer} \} = \{ 4 \} .$$

This statement reads "the set of m such that m + 3 = 7 , with the domain of m being the set of integers, is the set $\{ 4 \}$."

The " $\{$ m" denotes "the set of m," while the vertical line " | " denotes the phrase "such that."

Let's write another solution set in set builder notation:

$$\{ x \mid x - 5 = 10 \text{ , x real} \} = \{ 15 \} .$$

This statement reads "the set of x such that x - 5 = 10 , with the domain of x being the set of real numbers, is the set $\{ 15 \}$."

You might benefit by writing the set builder notation for:

> "the set of x such that x + 3 = 4 ,
> with the domain of x being the set
> of real numbers, is the set $\{ 1 \}$."

Write the set builder notation here:

The correct notation would be

$$\{ x \mid x + 3 = 4 \text{ , x real} \} = \{ 1 \} .$$

Now, suppose we considered an open sentence with two variables x and y :

$y = x + 2$, where the domain of x is the set of real numbers.

Let x have the value of -3 (i.e., set $x = -3$). The corresponding value of y is

$$y = (-3) + 2 = -3 + 2 = -1 \; .$$

That is, whenever $x = -3$, $y = -1$. We have the ordered pair $(-3, -1)$.

Suppose $x = -1$, then y is

$$y = (-1) + 2 = -1 + 2 = 1$$

That is, whenever $x = -1$, $y = 1$. Thus, we have the ordered pair $(-1, 1)$.

By continuing to assign values to x from the set of real numbers we could obtain more ordered pairs where the relationship between the x and its corresponding y is determined by the open sentence:

$y = x + 2$, domain of x is set of real numbers.

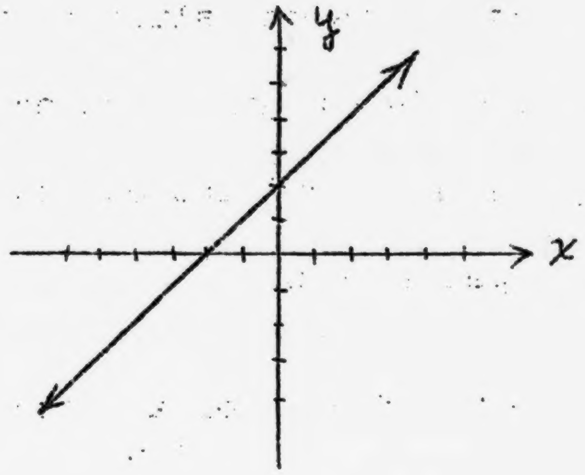
Hence, we could obtain a set of ordered pairs which can be described in set builder notation as

$$\{(x, y) \mid y = x + 2 , x \text{ real}\} .$$

This reads "the set of ordered pairs (x, y) such that $y = x + 2$, with the domain of x being the set of real numbers."

The statement $\{(x, y) \mid y = x + 2 , x \text{ real}\}$ specifies a set of ordered pairs which have the coordinate x and its corresponding coordinate y related to each other by the rule $y = x + 2$. That is, for every value of x selected the corresponding y is 2 greater than the value of x. We call such a statement $\{(x, y) \mid y = x + 2 , x \text{ real}\}$ a relation.

A graph of the relation is shown below:



The relation or its graph describes the solution set of the open sentence $y = x + 2$. Let's consider the solution set (i.e., the relation) of $y = 3 - x$, with the domain of x the real numbers. The solution set is the relation

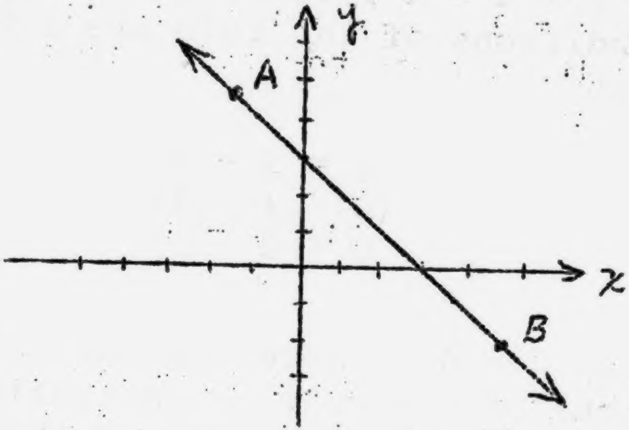
$$\{(x, y) \mid y = 3 - x \text{ , } x \text{ real}\} .$$

The solution set is a set of specific ordered pairs which satisfy the rule $y = 3 - x$. Since a graph consists of points on the plane which have specific coordinates described by ordered pairs, the solution set can be shown as a set of points on a graph. These sets of points may be rays, lines, half-planes, etc.

The solution set of the rule

$$y = 3 - x$$

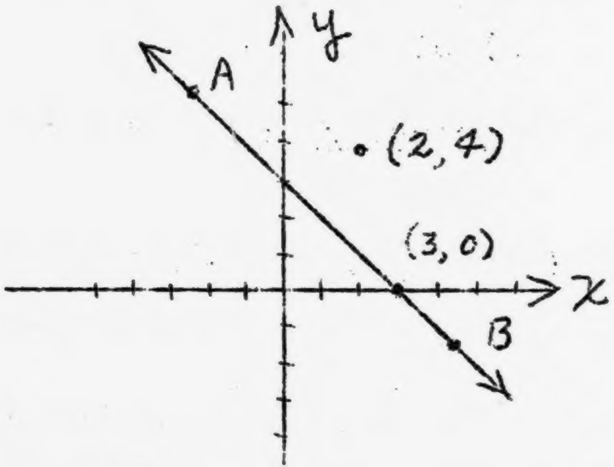
may be shown as the following graph:



What about the point named by the ordered pair (3, 0) , is it in the solution set of the rule

$$y = 3 - x \quad ?$$

The question can be answered by seeing if (3, 0) lies on the line $\overleftrightarrow{AB}$.



(3, 0) does lie on $\overleftrightarrow{AB}$, therefore (3, 0) is in the solution set of $y = 3 - x$. In other words, (3, 0) is an ordered pair in

$$\{(x, y) \mid y = 3 - x, x \text{ real}\}.$$

Let's check this out:

Substituting 3 for x and 0 for y in $y = 3 - x$, let's see if the conditions of the rule are satisfied:

$$y = 3 - x$$
$$(0) = 3 - (3)$$
$$0 = 0$$

Yes, $0 = 0$, therefore the conditions of the rule are satisfied and (3, 0) is a member of the solution set:

$$\{(x, y) \mid y = 3 - x, x \text{ real}\}$$

What about the point (2, 4)? It does not lie on AB, therefore we know it is not a member of the solution set. This is affirmed when we substitute into the rule $y = 3 - x$

$$\begin{aligned}(4) &= 3 - (2)\\ 4 &= 1\end{aligned}$$

This is not a true statement, therefore the conditions of the rule $y = 3 - x$ are not satisfied by the ordered pair (2, 4) and hence, (2, 4) is not a member of the solution set.

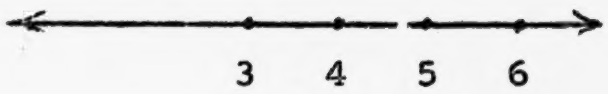
Before we go further in this direction, let's clarify what we mean by the symbols $=$, $>$, $<$, $\geq$, and $\leq$:

$=$	equal to
$>$	greater than
$<$	less than
$\geq$	greater than or equal to
$\leq$	less than or equal to.

Open number sentences in which these symbols are used have names which we should note:

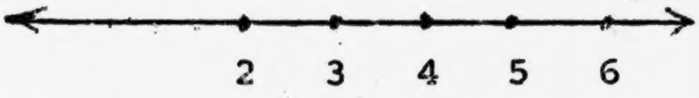
$x + 2 = 5$	equation
$y = x + 4$	equation
$x + 2 > 5$	inequality
$y > x + 4$	inequality
$x + 2 < 5$	inequality
$y < x + 4$	inequality
$x + 2 \geq 5$	inequality
$y \geq x + 4$	inequality
$x + 2 \leq 5$	inequality
$y \leq x + 4$	inequality

The graph of the solution set: $\{x \mid x > 2 \text{ , } x \text{ integer}\}$ for rule $x > 2$ is



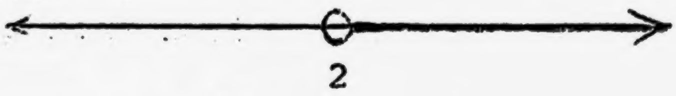
As far as you wish to extend the line, every integer larger than 2 would be shown as being in the solution set.

The solution set $\{x \mid x \geq 2 \text{ , } x \text{ integer}\}$ for rule $x \geq 2$ is

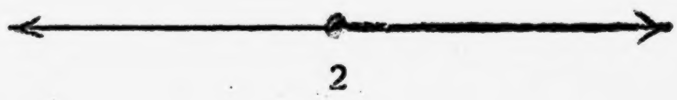


This time we have "x is greater than or equal to 2." Hence, 2 is included in the graph.

The solution set $\{x \mid x > 2 \text{ , } \underline{x \text{ real}}\}$ for rule $x > 2$ allows x to be selected from the set of real numbers. This means that every point on the number line to the right of 2 is included in the graph. Hence, we have the graph for $\{x \mid x > 2 \text{ , } x \text{ real}\}$:

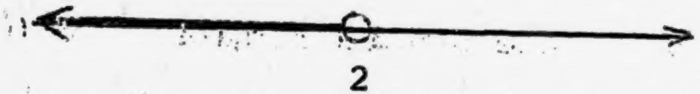


Since $\{x \mid x \geq 2 \text{ , } x \text{ real}\}$ includes the 2, we have for $\{x \mid x \geq 2 \text{ , } x \text{ real}\}$:

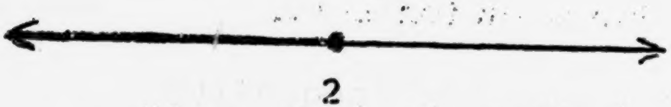


Similarly, we have the graphs for the following:

i. $\{x \mid x < 2 \text{ , } x \text{ real}\}$



ii. $\{x \mid x \leq 2 \text{ , } x \text{ real}\}$

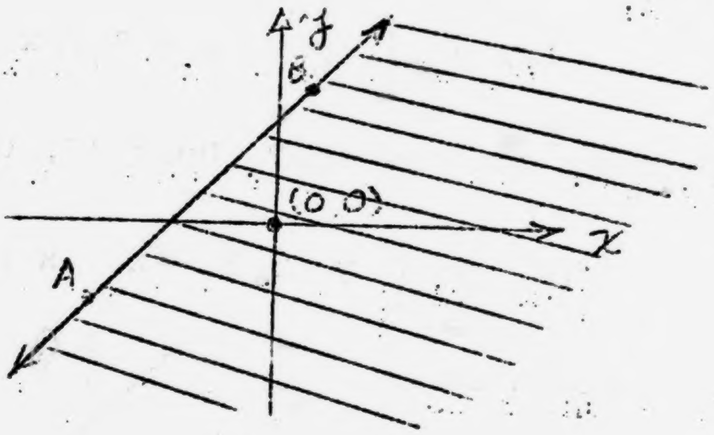


In (i.) above, is the point 2 a member of the solution set? No, 2 is excluded. What about the point 1 ? 1 is less than 2 and thus is a member of the solution set.

Let's now look at some graphs of solution sets (i.e., relations) on a plane which involve inequalities. The graph of

$$\{(x, y) \mid y \leq x + 2 \text{ , } x \text{ real}\}$$

is



What about the point (0, 0) at the origin, is it a member of the solution set? Does it satisfy the conditions of the relation

$$\{(x, y) \mid y \leq x + 2 \text{ , } x \text{ real}\} \text{ ?}$$

We can determine in two different ways whether or not (0, 0) is a member of the solution set.

1. Look at the graph. Is (0, 0) on the line $\overleftrightarrow{AB}$ or in the shaded region? If (0, 0) satisfies either of these conditions, it is a member of the solution set. If (0, 0) is not on $\overleftrightarrow{AB}$ or in the shaded region, it is not a member of the solution set.

2. Consider the inequality

$$y \leq x + 2 .$$

Does (0, 0) satisfy the inequality?
Substituting:

$$0 \leq 0 + 2$$
$$0 \leq 2$$

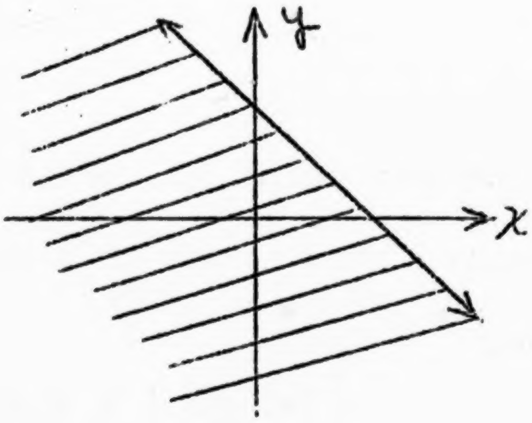
Yes, 0 is less than or equal to 2. Therefore, (0, 0) does satisfy the inequality and, hence, is a member of

$$\{(x, y) \mid y \leq x + 2 , x \text{ real}\} .$$

Why don't you try one? Does (0, 0) satisfy

$$\{(x, y) \mid y \leq 3 - x , x \text{ real}\} ?$$ Yes__;No__.

The graph is shown as



Did you check Yes? You are correct, because

1. (0, 0) is in the shaded area.

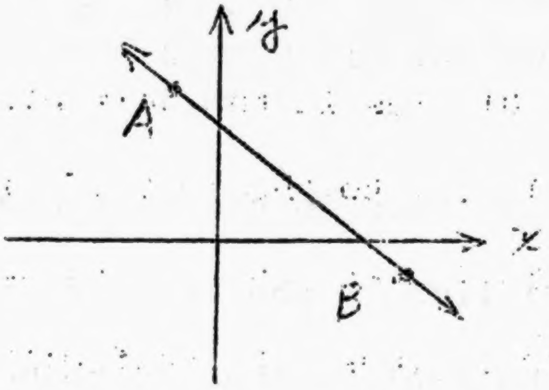
and 2. For $y \le 3 - x$
$0 \le 3 - 0$
$0 \le 3$ is true.

Let's look at a different problem. Suppose you had only the graph of the line for the relation

$$\{(x, y) \mid \le 3 - x \text{ , } x \text{ real}\}$$

and you wanted to know what region should be shaded. How would you determine the answer?

Consider the graph for $\{(x, y) \mid y \le 3 - x \text{ , } x \text{ real}\}$



with the shading deleted. Where should you shade?

**Read Appendix A-2 (the pink folder).
The remainder of this lesson will
assume you have read Appendix A-2.**

The key sentence in Appendix A-2 is: IF ONE POINT ON THE HALF-PLANE IS TO THE LEFT OF LINE $\overleftrightarrow{MN}$, THEN ALL THE POINTS OF THE HALF-PLANE ARE TO THE LEFT OF LINE $\overleftrightarrow{MN}$. This fact will help us to answer the question concerning what region should be shaded in the above figure.

The origin (0, 0) lies to the left of the line $\overleftrightarrow{AB}$ in the above figure. Does the origin lie in the half-plane that is to be shaded? If the origin lies in the half-plane that should be shaded, then the origin (0, 0) is a member of the solution set

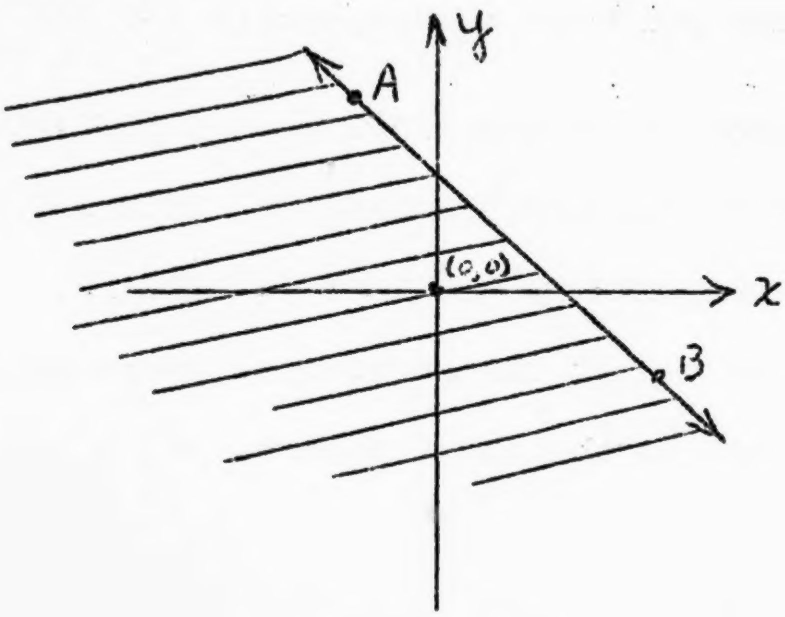
$$\left\{ (x, y) \;\middle|\; y \leq 3 - x \text{ , } x \text{ real} \right\} .$$

That is, if (0, 0) satisfies the conditions of the rule $y \leq 3 - x$, then (0, 0) will lie in the shaded region of the graph.

Let's check:

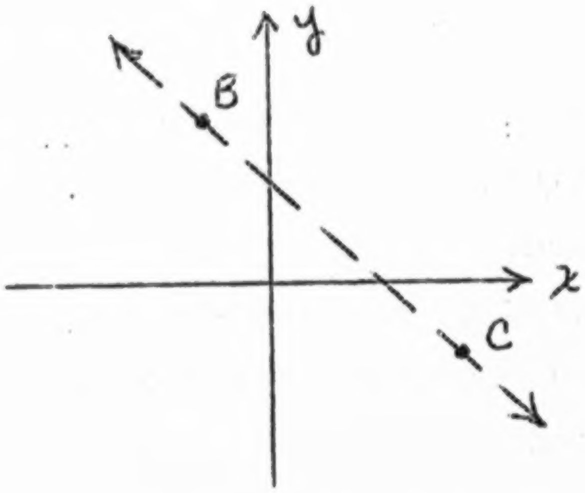
$$\text{substituting } (0, 0), \quad \begin{aligned} y &\leq 3 - x \\ 0 &\leq 3 - 0 \\ 0 &\leq 3 \end{aligned} \quad \text{is true.}$$

Therefore, (0, 0) lies in the half-plane that is to be shaded. (0, 0) lies to the left of line $\overleftrightarrow{AB}$. Hence, all the points of the half-plane that is to be shaded are to the left of line $\overleftrightarrow{AB}$. Thus, the half-plane to be shaded is to the left of $\overleftrightarrow{AB}$:



Let's look at the graph of the relation

$\{(x, y) \mid y > 2 - x\ ,\ x \text{ real}\}$ where the shaded region is deleted:



$\overleftrightarrow{BC}$ is dashed because the symbol $>$ is used. If $\geq$ were used, $\overleftrightarrow{BC}$ would be solid. What side of line $\overleftrightarrow{BC}$ should be shaded to obtain the complete graph of

$$\{(x, y) \mid y > 2 - x\ ,\ x \text{ real}\}\ ?$$

Does the origin satisfy the rule $y > 2 - x$?

$$\begin{array}{ll} & y > 2 - x \\ \text{substituting } (0,\ 0)\ , & 0 > 2 - 0 \\ & 0 > 2 \qquad \text{is not true.} \end{array}$$

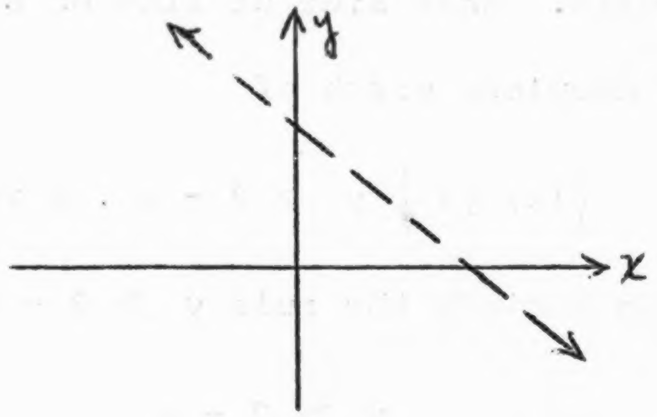
Therefore, the origin does not satisfy the rule $y > 2 - x$. (0, 0) is not a member of the solution set $\{(x, y) \mid y > 2 - x\ ,\ x \text{ real}\}$. Hence, (0, 0) is not in the shaded region of the graph. Note: recall that the lines and the shaded regions of a graph are the set of points which satisfy the relation $\{(x, y) \mid y > 2 - x\ ,\ x \text{ real}\}$. Since the origin (0, 0) is not in the shaded region of the graph and the origin is to the left of line $\overleftrightarrow{BC}$, then the shaded region of the graph will be to the RIGHT of line $\overleftrightarrow{BC}$.

Here are two for you to shade:

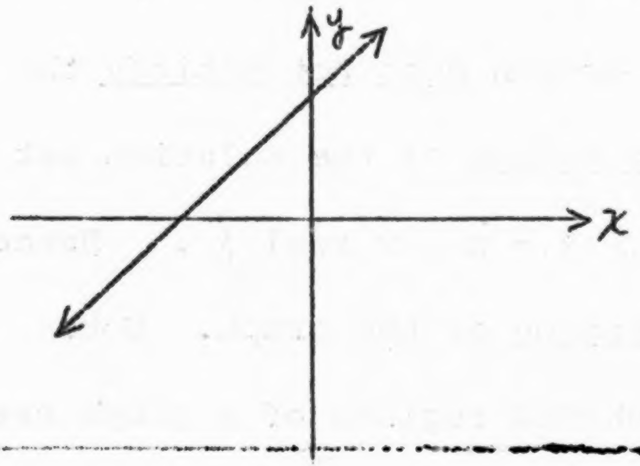
GIVEN the graphs of the following two relations with the shaded regions deleted;

IDENTIFY, by shading, the half-plane or regions which satisfy the inequality for each relation.

1. $\left\{ (x, y) \mid y < 4 - x \text{ , } x \text{ real} \right\}$



2. $\left\{ (x, y) \mid y \geq x + 3 \text{ , } x \text{ real} \right\}$



The solutions of these two problems are shown on page 21 .

The two vertical lines $|-5|$ mean that the negative sign is to be changed to a + sign. That is, $|-5| = +5$. We call the value $|-5|$, the absolute value of -5 . Hence, the absolute value of -5 is +5 . The absolute value of +5 is +5 . That is, $|5| = 5$. Similarly,

$$|0 - 3| = |-3| = 3$$
$$|0 + 2| = |2| = 2$$
$$|7| = 7$$
$$|-7| = 7$$

We'll discuss the absolute value concept more in Lesson VII-b.

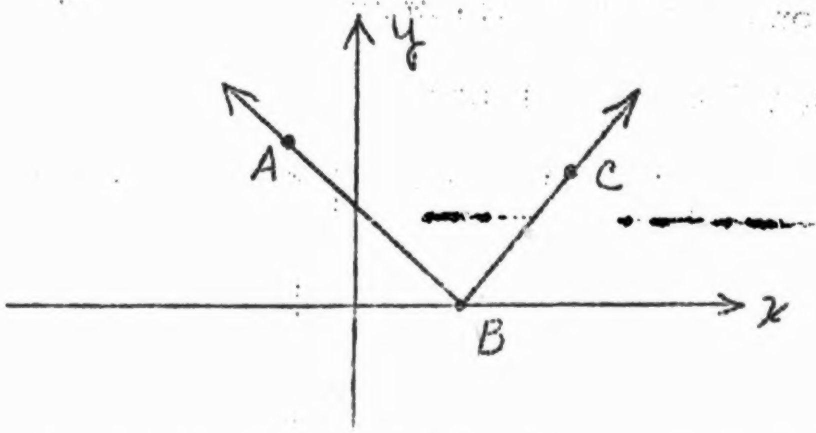
If we have $|x + 3|$ and $x = 0$, then

$$|x + 3| = |0 + 3| = |3| = 3 .$$

In like manner, if we have $|x - 4|$ and $x = 0$, then

$$|x - 4| = |0 - 4| = |-4| = 4 .$$

The graph for the relation $\{(x, y) \mid y \leq x - 2 ,$ $x \text{ real}\}$ with the shaded region deleted is shown below:



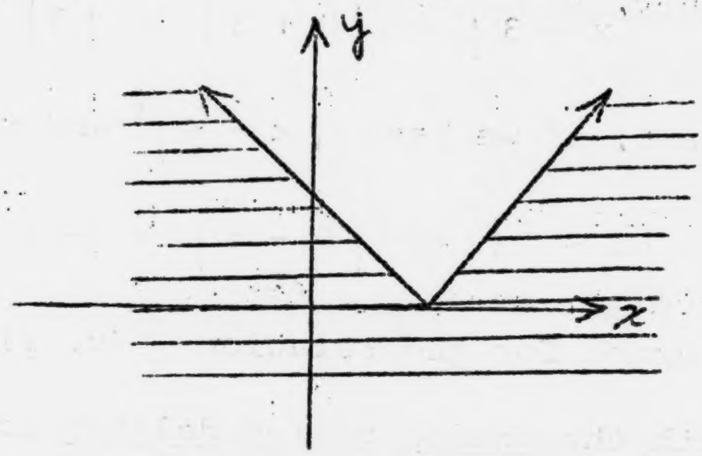
What region should be shaded? Again, we can determine the location of the shaded region by determining whether or not the origin (0, 0) satisfies the rule $y \leq |x - 2|$. Substituting (0, 0) in the rule we have

$$y \leq |x - 2|$$
$$0 \leq |0 - 2|$$
$$0 \leq |-2|$$

thus

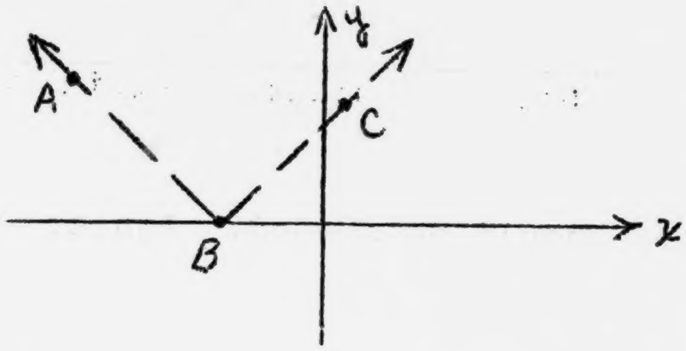
$$0 \leq 2 \quad \text{is true.}$$

Therefore, the origin (0, 0) is in the shaded region. Hence, all points on the same side of the ∠ ABC as the origin are in the shaded region. The complete graph is



Consider the following relation and its graph with the shaded region deleted:

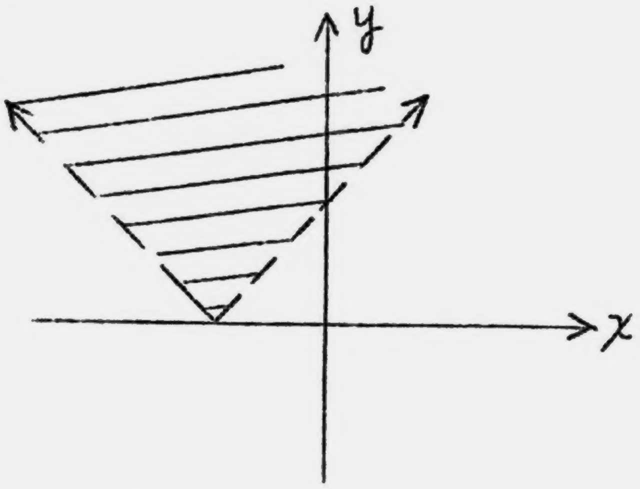
$$\{(x, y) \mid y > |x + 2| , x \text{ real}\}$$



Checking the origin (0, 0) , we note that

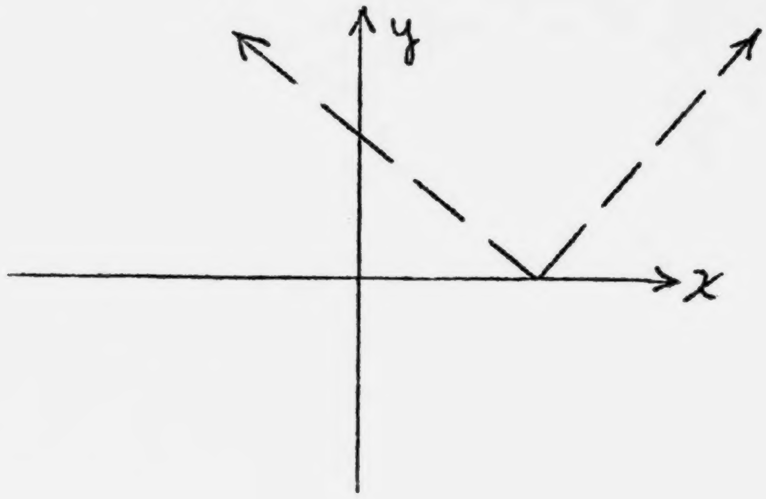
$$y > |x + 2|$$
$$0 > |0 + 2|$$
$$0 > |2|$$
$$0 > 2 \quad \text{is \underline{not true}.}$$

Therefore, the origin is not in the shaded region. Hence, the shaded region is on the opposite side of the ∠ ABC from the origin. That is, the shaded region is the interior of ∠ ABC. We have

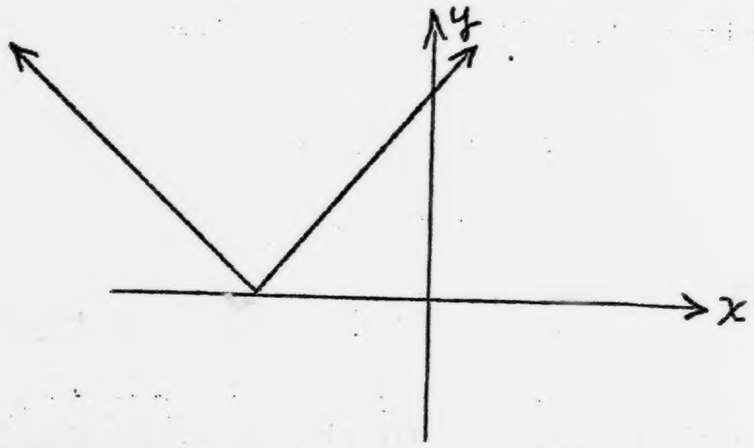


For your practice, shade the appropriate regions for these two relations:

3. $\left\{(x, y) \,\middle|\, y < |x - 4| \;, \; x \text{ real}\right\}$



4. $\{(x, y) \mid y \geq |x + 5| \text{ , x real}\}$



The correct shading of these graphs are shown on page 21 .

Answers to Performance Tasks

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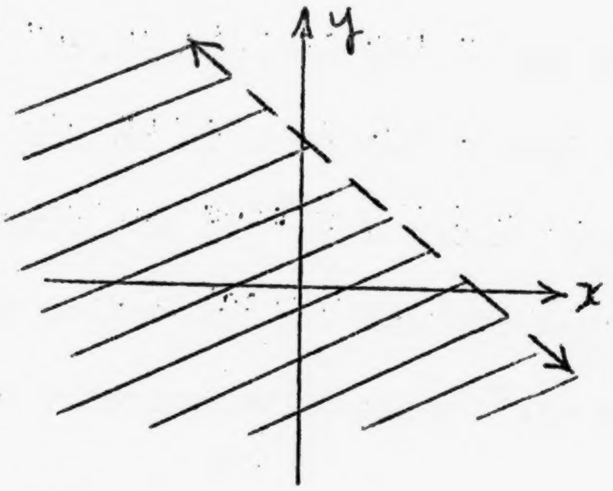
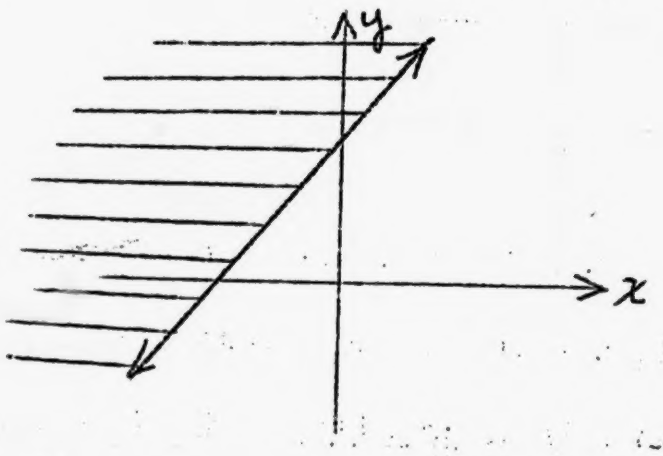
1.



2.



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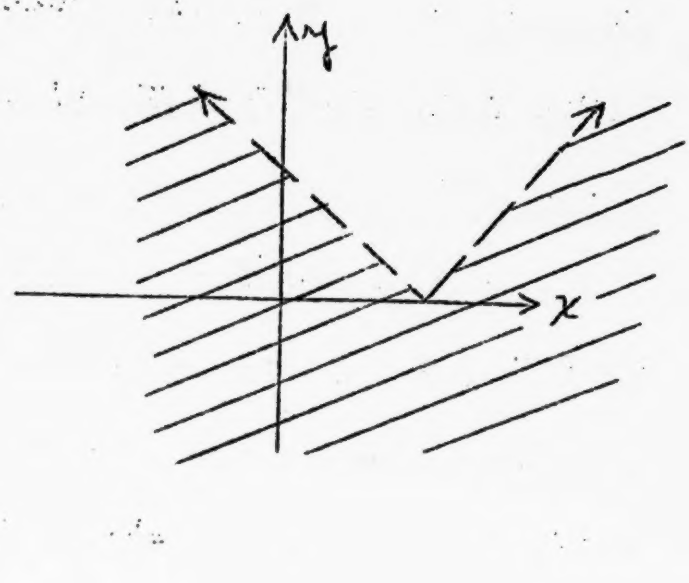
3.



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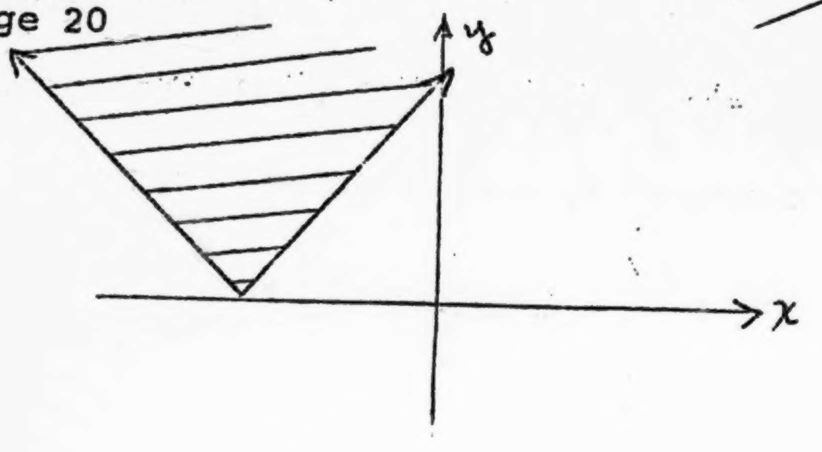
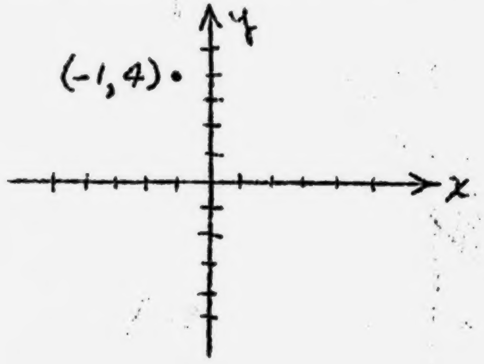
4.



VII-b. CONSTRUCTING GRAPHS

We have discussed ordered pairs (x, y) in previous lessons. For each value of x in an ordered pair, there is a corresponding value of y. In the ordered pair (-1, 4) the value of x is -1 and the corresponding value of y is 4 . Ordered pairs are the coordinates of points on a Cartesian plane:



Equations like y = x + 5 have sets of ordered pairs which satisfy the conditions of the equation. For the equation y = x + 5 , there is a specific set of ordered pairs which satisfy the equation. That is, there is a solution set of ordered pairs for y = x + 5 . The relation $\{(x, y) \mid y = x + 2, x \text{ real}\}$ describes the solution set of y = x + 2 . The solution set is determined by computing corresponding values of y for different selected values of x . For the equation y = x + 5 , let x = 0 . The corresponding value of y can be computed by substituting 0 for x in the equation

$$\begin{aligned} y &= x + 5 \\ y &= 0 + 5 \\ y &= 5 \end{aligned}$$

Thus, when $x = 0$, $y = 5$. That is, (0, 5) is a member of the solution set described by the relation $\left\{(x, y) \mid y = x + 5 \text{ , } x \text{ real}\right\}$.

If $x = -2$, let's compute the corresponding value of y for $y = x + 5$:

$$\begin{aligned} y &= x + 5 \\ y &= -2 + 5 \\ y &= 3 \end{aligned}$$

Thus, (-2, 3) is also a member of the solution set of the relation $\left\{(x, y) \mid y = x + 5 \text{ , } x \text{ real}\right\}$.

Consider the equation $y = -3 - x$. Let $x = 0$:

$$\begin{aligned} y &= -3 - x \\ y &= -3 - 0 \\ y &= -3 \end{aligned}$$

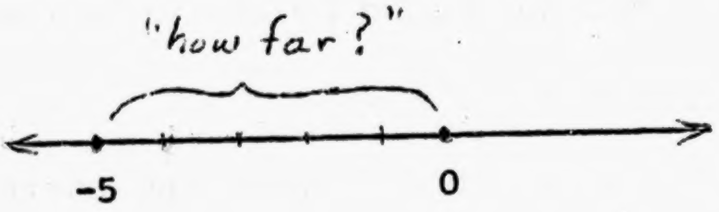
(0, -3) is therefore a member of the solution set of the relation $\left\{(x, y) \mid y = -3 - x \text{ , } x \text{ real}\right\}$. For $x = -4$:

$$\begin{aligned} y &= -3 - x \\ y &= -3 - (-4) \\ y &= -3 + 4 \\ y &= 1 \end{aligned}$$

(-4, 1) is a member of the solution set of $\left\{(x, y) \mid y = -3 - x \text{ , } x \text{ real}\right\}$, also.

Some equations may involve absolute values; for example, $y = |x + 6|$. We mentioned in Lesson VII-a that the absolute value of a number like -5 was +5 . That is, $|-5| = 5$. Also, $|+7| = +7$. The absolute value of a number refers to the distance of that number from 0 .

Consider the location of the point representing -5 on the number line:



How far is -5 from 0 ? __________ . Did you write "5" or "5 units"? You are correct. We certainly don't say, "Joe is standing -5 feet to the left of the table and Jack is standing 7 feet to the right of the table." Distance is always given in terms of a positive number. When we are working with distance we are interested in the absolute number of units from one point to another point. Direction is of no consequence. The distance from -5 to 0 is the same as the distance from 0 to -5 . The absolute value symbol is used to denote distance. Thus, $|-5| = +5$ and $|+5| = +5$.

Consider the equation $y = |x - 4|$ and let $x = -3$. What is the corresponding value of y ? Computing for y :

$$y = |x - 4|$$

substituting -3 for x , $y = |-3 - 4|$

$$y = |-7|$$

$$y = 7$$

(-3, 7) is a member of the set of ordered pairs which satisfy the relation $\{(x, y) \mid y = |x - 4| \text{ , } x \text{ real}\}$.

Let $x = 6$ for $y = |x - 4|$:

$$y = |x - 4|$$
$$y = |6 - 4|$$
$$y = |2|$$
$$y = 2$$

For each of the following equations compute the corresponding values of y for the values of x given (show your calculations):

1. Compute y from $y = x - 7$,

   a. when $x = -4$

   b. when $x = 0$

   c. when $x = 3$

2. Compute y from $y = 5 - x$,

   a. when $x = -3$

   b. when $x = 0$

   c. when $x = 3$

3. Compute y from $y = |x - 2|$,

   a. when $x = -2$

   b. when $x = 0$

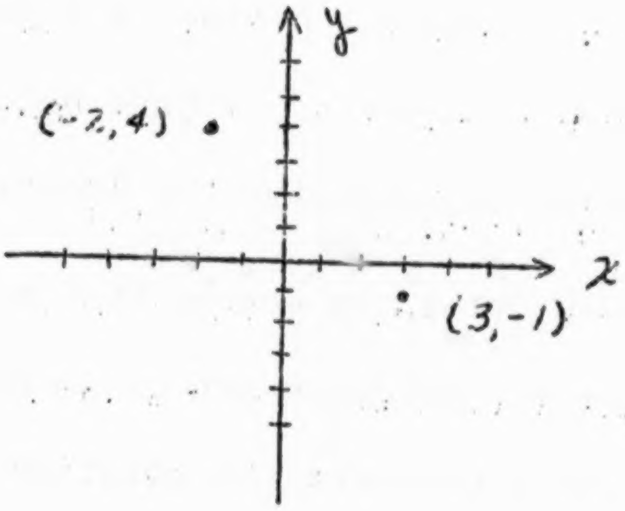
c. when x = 2

d. when x = 4

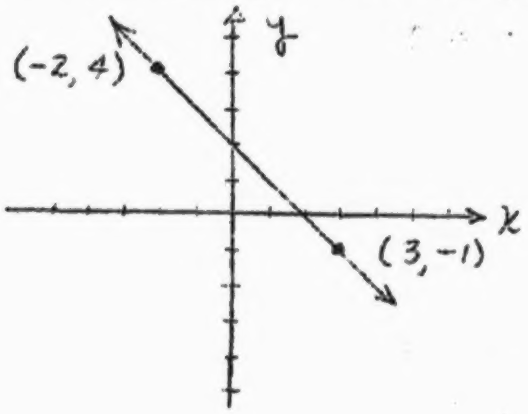
e. when x = 6

Solutions to these problems are found on page 29 .

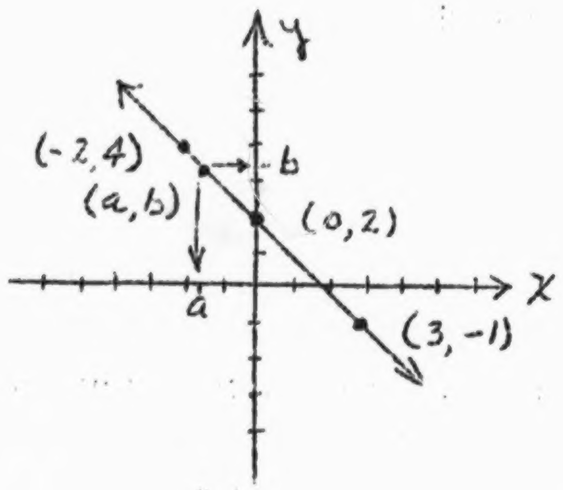
In Lesson II in our discussion on the line, we noted that any two points define the location of a line. Suppose we denote two points on a Cartesian plane by their ordered pairs:



Next, let's draw a line through the two points:



Hence, knowing two ordered pairs we have been able to draw a straight line on a Cartesian plane. Of course, there are also other points on the line. Notice that the point (0, 2) is on the line:



The point (a, b) is noted here to illustrate that there can be points on a line in a plane whose coordinates are real numbers but not integers. A line on a plane includes all real values of x . However, we have noted that only two points are needed to establish the location of a line.

In Lesson VII-a, we stated that a relation or its graph describes the solution set of an equation or an inequality. Let's consider the solution set of $y = 2 - x$, with the domain of x being the real numbers. The solution set is described by the relation $\{(x, y) \mid y = 2 - x \text{ , } x \text{ real}\}$. In order to describe the solution set of y = 2 - x with a graph, we must plot the points of ordered pairs which satisfy the relation $\{(x, y) \mid y = 2 - x \text{ , } x \text{ real}\}$.

Since the domain of x is the set of real numbers, we can choose x to be -3 . What is the corresponding value of y ? Substituting -3 for x , we have:

$$\begin{aligned} y &= 2 - x \\ y &= 2 - (-3) \\ y &= 2 + 3 \\ y &= 5 \end{aligned}$$

That is, (-3, 5) is a member of the solution set.

Letting x = -2 , we have:

$$\begin{aligned} y &= 2 - x \\ y &= 2 - (-2) \\ y &= 2 + 2 \\ y &= 4 \end{aligned}$$

(-2, 4) is also a member of the solution set.

Perhaps a chart or table might help us to organize our findings:

x	-3	-2
y	5	4

The table of ordered pairs shows that when x = -3 , y = 5 and when x = -2 , y = 4 .

Let's determine more ordered pairs in the solution set and place them in the table of ordered pairs. For $x = -1$:

$$y = 2 - x$$
$$y = 2 - (-1)$$
$$y = 2 + 1$$
$$y = 3$$

Hence,

x	-3	-2	-1
y	5	4	3

For $x = 0$:

$$y = 2 - x$$
$$y = 2 - 0$$
$$y = 2$$

Hence,

x	-3	-2	-1	0
y	5	4	3	2

For $x = 1$:

$$y = 2 - x$$
$$y = 2 - 1$$
$$y = 1$$

Hence,

x	-3	-2	-1	0	1
y	5	4	3	2	1

For x = 2 :

$$y = 2 - x$$
$$y = 2 - 2$$
$$y = 0$$

Hence,

x	-3	-2	-1	0	1	2
y	5	4	3	2	1	0

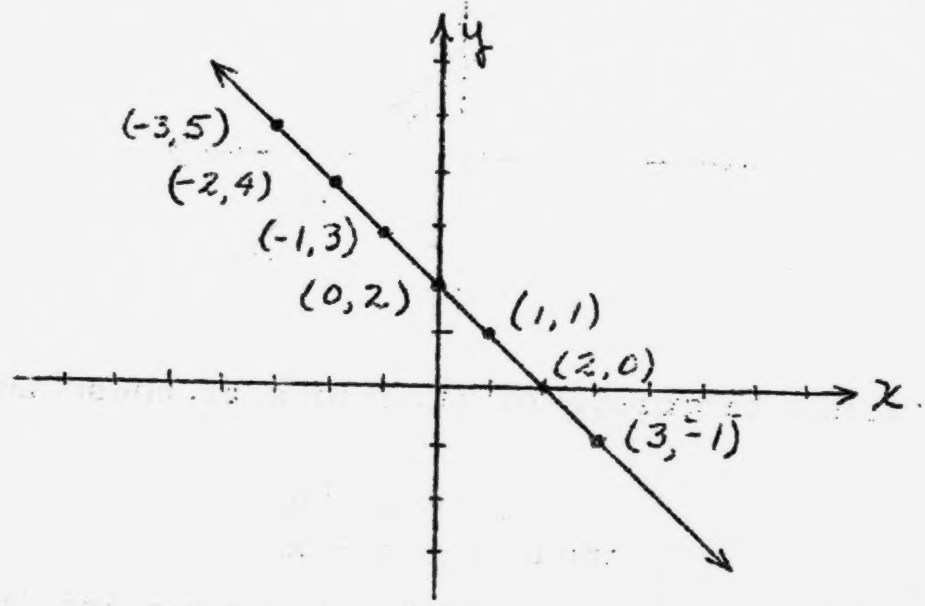
For x = 3 :

$$y = 2 - x$$
$$y = 2 - 3$$
$$y = -1$$

Hence,

x	-3	-2	-1	0	1	2	3
y	5	4	3	2	1	0	-1

Plotting the ordered pairs in the table, we have:

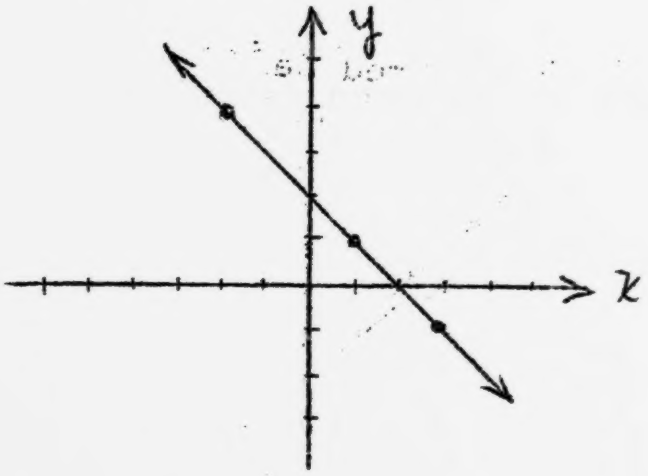


The line is drawn solid because the domain of x is the set of real numbers. Since the graph of the relation $\{(x, y) \mid y = 2 - x \text{ , } x \text{ real}\}$ is a line, only two ordered pairs would have been required. Perhaps a third ordered pair would have been helpful to check the accuracy of the other two ordered pairs. That is, if all three points computed do not lie on a straight line, we know that at least one of the ordered pairs was incorrectly computed.

Selecting any three values of x we could have computed the corresponding values of y and obtained a table of ordered pairs. If we had selected $x = -2$, $x = 1$, and $x = 3$, we would have had the table:

x	-2	1	3
y	4	1	-1

and the graph:



Since the graph of equations of these two types

$$y = x + b$$
$$\text{and} \quad y = c - x$$

(Note: b and c are integers)

are always straight lines, we can plot their graphs by computing the ordered pairs for only three points for each equation.

Let's construct the table of ordered pairs from which the graph of the relation $\{(x, y) \mid y = x + 3 , x \text{ real}\}$ could be constructed. Since any three points are sufficient to plot the straight line, we can select any three values of x . Let's choose x = -4 , x = -2 , x = 0 and compute the corresponding y's to complete the table:

x	-4	-2	0
y			

$y = x + 3$
$y = -4 + 3$
$y = -1$

$y = x + 3$
$y = -2 + 3$
$y = 1$

$y = x + 3$
$y = 0 + 3$
$y = 3$

Hence, we have the table of ordered pairs:

x	-4	-2	0
y	-1	1	3

Notice, that for convenience and ease in plotting the values of x start from some low value and increase in value by the same amount; i.e.: -4 to -2 increase of 2 units
-2 to 0 increase of 2 units

Here are several relations for you to practice constructing tables of ordered pairs from which the graph of each relation could be constructed:

4. $\{(x, y) \mid y = x - 3 , x \text{ real}\}$

x			
y			

5. $\{(x, y) \mid y = x - 5, \ x \text{ real}\}$

x			
y			

6. $\{(x, y) \mid y = 3 - x, \ x \text{ real}\}$

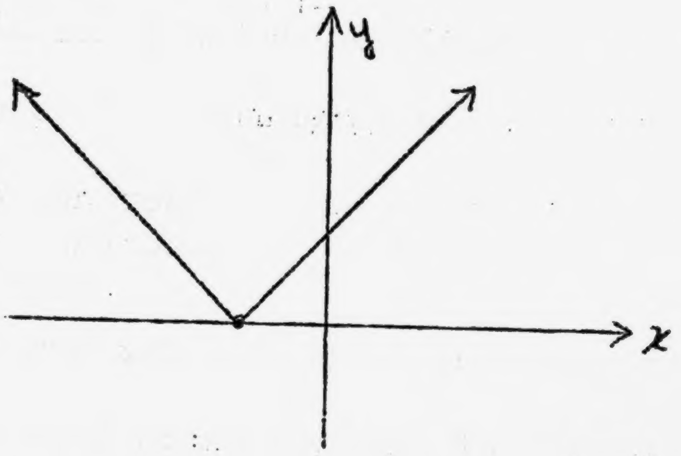
x			
y			

7. $\{(x, y) \mid y = -2 - x, \ x \text{ real}\}$

x			
y			

Solutions for these problems are found on page 29 .

You recall in Lesson VII-a that we worked with graphs which had the shape of an angle:



The graph of an equation involving an absolute value always has this angle shape. Notice that there are no values of y which are negative.

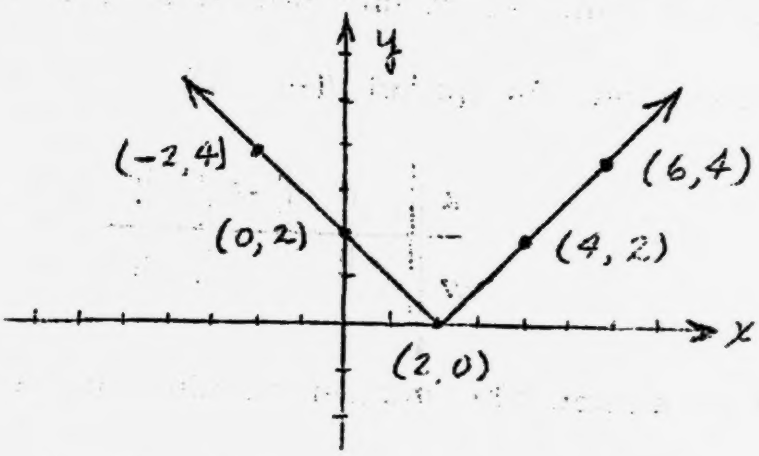
Let's check that statement by seeing what the graph of the relation $\left\{(x, y) \mid y = |x - 2|, \text{ x real}\right\}$ looks like. Selecting values for x we have the incomplete table.

x	-2	0	2	4	6
y					

Computing the corresponding values of each y like you did in your practice work on page 5, we have

x	-2	0	2	4	6
y	4	2	0	2	4

Plotting the points from the table, we have the graph:



Notice that in the above graph the three points on each ray, including a common point (2, 0) at the vertex of the angle, were a sufficient number of points (total of five) to draw the graph. The graph of the equation $y = |x - 2|$ is solid rather than just the points corresponding to the table of ordered pairs because the domain of x is the set of real numbers. The value of y at the vertex of the angle will always be 0 for such an equation as $y = |x - 2|$.

Let's construct the table for the relation $\{(x, y) \mid y = |x + 3|, x \text{ real}\}$. Only five points are necessary, so we will select only five values of x. However, since we've stated that the value of y at the vertex of the graph will always be 0 , then let's first choose a value of x whose corresponding value of y is 0 . Letting x = -3 will assure that y = 0 :

$$\begin{aligned} y &= |x + 3| \\ y &= |-3 + 3| \\ y &= |0| \\ y &= 0 \end{aligned}$$

Therefore, we can begin the construction of the table of ordered pairs at the point (-3, 0) :

x			-3		
y			0		

Notice that since the vertex of the angle is the center point of the graph, we place the ordered pair (-3, 0) in the center position in the table. Moving in equal distance steps to the left and to the right of x = -3 , we have

x	-7	-5	-3	-1	1
y			0		

Computing the corresponding values of y :

$y = |x + 3|$
$y = |-7 + 3|$
$y = |-4|$
$y = 4$

$y = |x + 3|$
$y = |-5 + 3|$
$y = |-2|$
$y = 2$

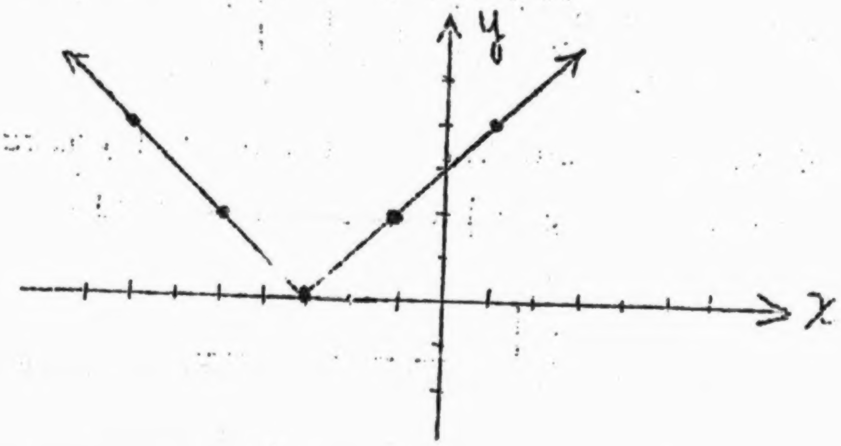
$y = |x + 3|$
$y = |-1 + 3|$
$y = |2|$
$y = 2$

$y = |x + 3|$
$y = |1 + 3|$
$y = |4|$
$y = 4$

The completed table is

x	-7	-5	-3	-1	1
y	4	2	0	2	4

and the graph looks like



Construct the table of values for the relation

$\{(x, y) \mid y = |x - 4|, x \text{ real}\}$:

x					
y					

Remember that ALL VALUES OF y WILL BE EITHER 0 OR A POSITIVE NUMBER. If your computations for y yield a negative value for y , you have made an error. Except for the center point, you may use any value of x in the table. One correct table using equal steps of two from the value of x at the vertex is

x	0	2	4	6	8
y	4	2	0	2	4

Here are several other relations for your practice:

8. Construct the table of ordered pairs for the relation $\{(x, y) \mid y = |x - 1| , x \text{ real}\}$

x					
y					

9. Construct the table of ordered pairs for $\{(x, y) \mid y = |x + 2| , x \text{ real}\}$

x	
y	

10. Construct the table of ordered pairs for $\{(x, y) \mid y = |x - 3| , x \text{ real}\}$

x	
y	

Solutions to these problems are shown on page 30 .

When a relation involves an inequality, we use the table of ordered pairs to graph the edge of the region which is to be shaded. Consider the relation $\{(x, y) \mid y \geq |x + 3|, x \text{ real}\}$. In order to construct the graph of this relation we perform the following three steps:

i. Construct the table of ordered pairs of the equation $y = |x + 3|$.

ii. Plot the ordered pairs from the table to locate the edge of the shaded region of the graph of the inequality.

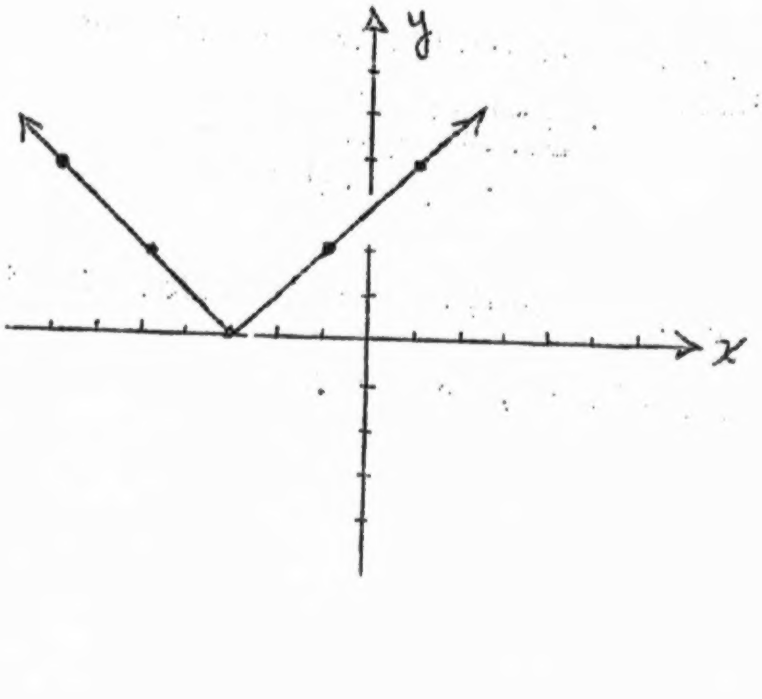
iii. Determine as we did in Lesson VII-a whether or not the origin is in the shaded region; shade the appropriate region.

Let's go through each of these steps for the relation $\{(x, y) \mid y \geq |x + 3|, x \text{ real}\}$:

i.

x	-7	-5	-3	-1	1
y	4	2	0	2	4

ii.



iii.

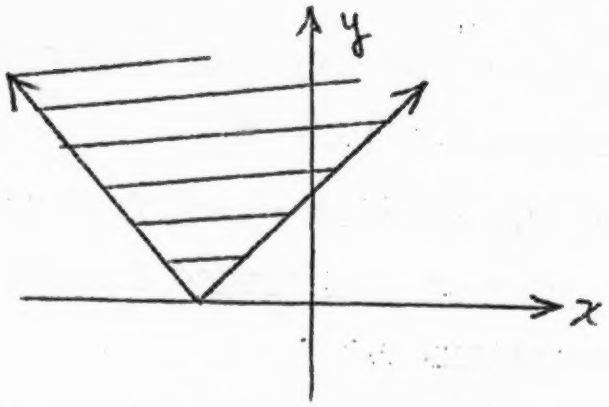
$y \geq |x + 3|$

substituting (0, 0) , $0 \geq |0 + 3|$

$0 \geq |+3|$

$0 \geq 3$ is not true .

Therefore, we have



If you were asked to construct the table of ordered pairs from which the graph of the relation $\{(x, y) \mid y < |x - 1|, x \text{ real}\}$ could be constructed, you would perform step (i.) as follows:

Set: $y = |x - 1|$

Choose the values of x :

x	-3	-1	1	3	5
y			0		

Complete the construction of the table:

x	-3	-1	1	3	5
y	4	2	0	2	4

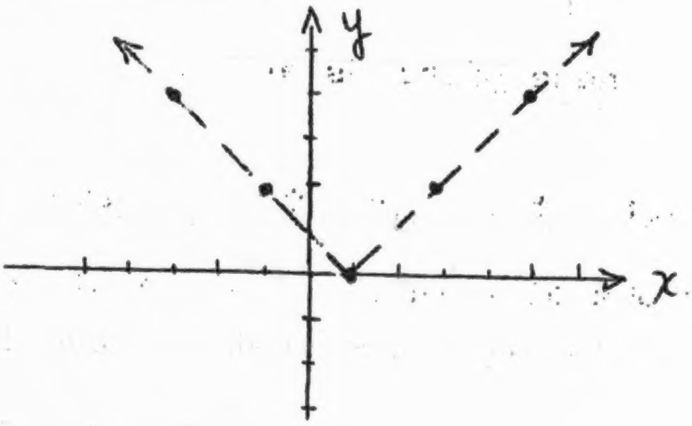
Notice step (i.) is the same whether the inequality is of the form ≤ , ≥ , < , or > .

Later in this lesson you will be asked to construct a graph for a relation similar to $\left\{ (x, y) \;\middle|\; y < |x - 1|, \ x \text{ real} \right\}$. Steps (i.), (ii.), and (iii.) should be followed:

i. We have already constructed the table.

x	-3	-1	1	3	5
y	4	2	0	2	4

ii. Since we have $<$, the edge of the shaded region will be dashed:



iii. Checking the origin (0, 0)

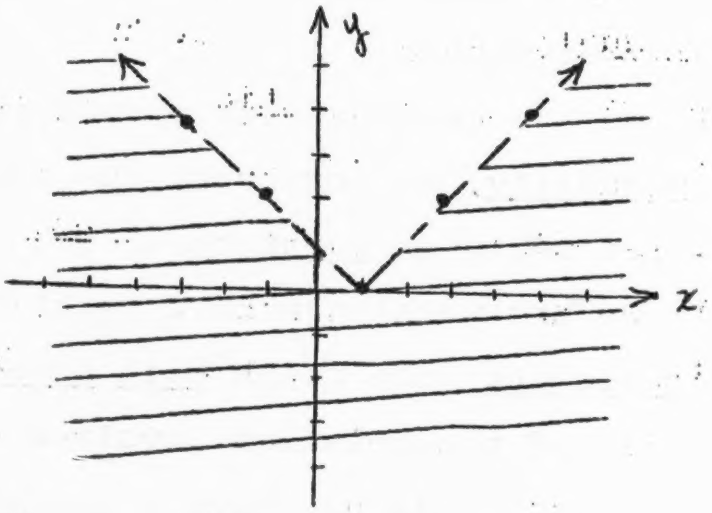
$$y < |x - 1|$$

substituting (0, 0), $0 < |0 - 1|$

$$0 < |-1|$$

$0 < 1$ is true

Hence, we have



Construct the table of ordered pairs from which the graph of each of the two following relations could be constructed:

11. $\{(x, y) \mid y \geq |x + 2|, x \text{ real}\}$

x	
y	

12. $\{(x, y) \mid y > |x - 4|, x \text{ real}\}$

x	
y	

Solutions for these problems are shown on page 30 .

Now, let's try bring what we have learned about constructing graphs of relations into a clear focus. We know that:

i. If we are working with an inequality with $\leq$ or $\geq$, the edge of the shaded region on the plane will be solid;

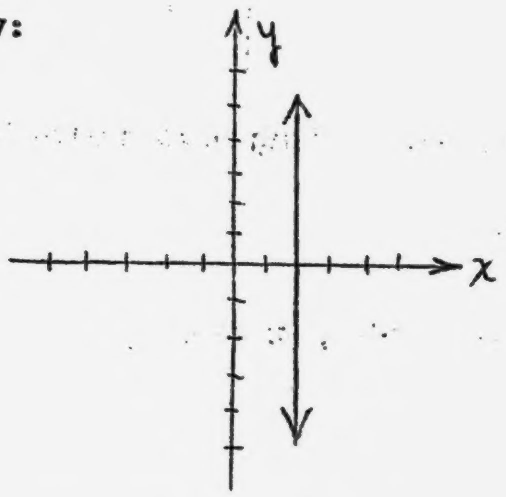
ii. If we are working with an inequality with $<$ or $>$, the edge of the shaded region on the plane will be dashed;

iii. If we are working with a relation involving an inequality, we construct the table of ordered pairs using an equation;

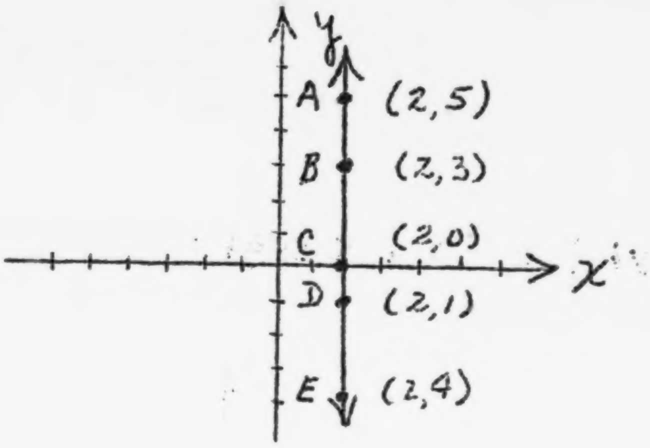
iv. If we are working with a relation involving an inequality, the graph will have a shaded region, while if the relation involves only an equation, the graph will not have a shaded region;

v. If we are working with a relation involving an inequality, the location of the shaded region is determined by whether or not the origin (0, 0) satisfies the conditions of the inequality.

There are some relations for which it is not necessary to construct tables of ordered pairs in order to construct their graphs. Consider the line shown on the Cartesian plane shown below:



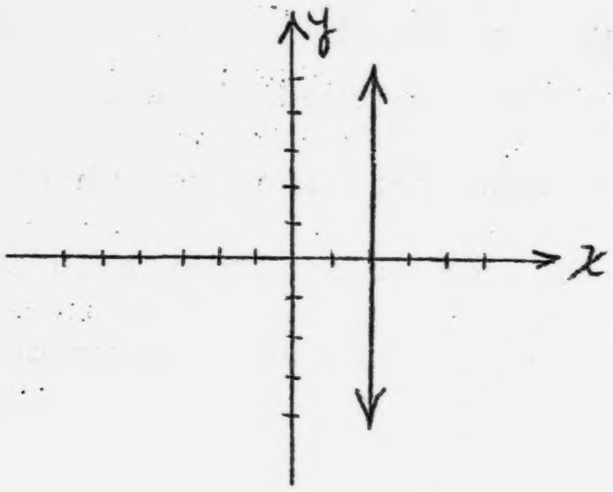
Now, let's notice the coordinates of some points on the line.



The value of x for every point is x = 2 . It doesn't seem to matter what the value of y is for points on the line, the x = 2 . Hence, the line is the graph of the relation

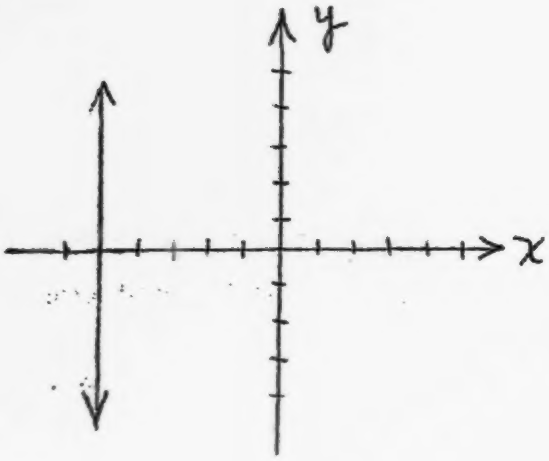
$\{(x, y) \mid x = 2 , x \text{ real}\}$.

Given the relation $\{(x, y) \mid x = 2 , x \text{ real}\}$, the graph is

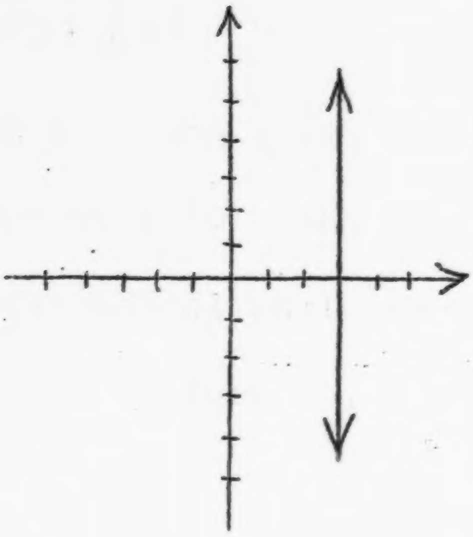


The graphs of the following two relations are constructed in the same manner:

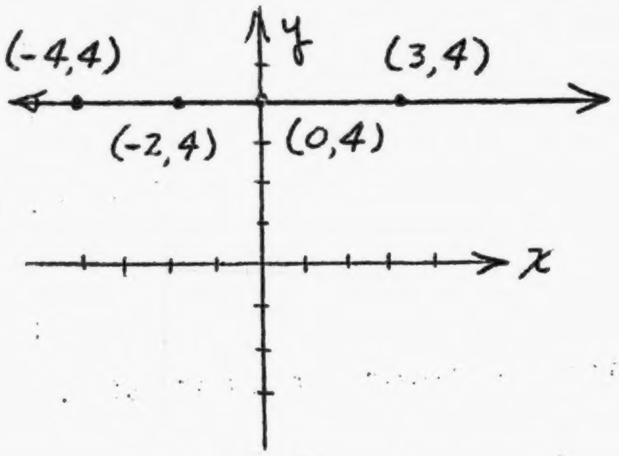
$\{(x, y) \mid x = -5 , x \text{ real}\}$



$\{(x, y) \mid x = 3 , x \text{ real}\}$



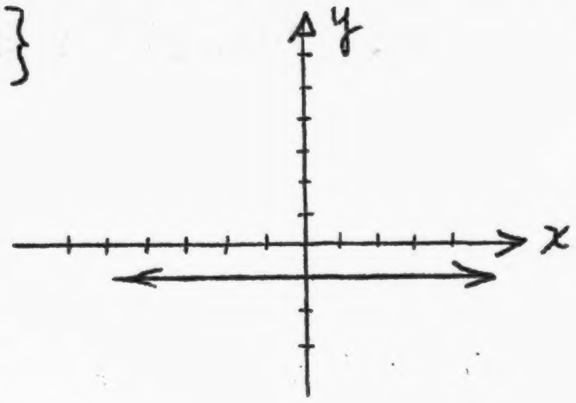
Consider the graph of the line shown below and notice the coordinate points on the line:



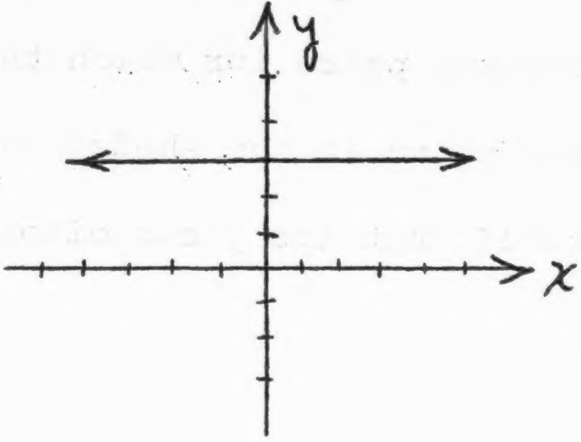
For every point on the line, $y = 4$. Hence, the line is the graph of the relation $\{(x, y) \mid y = 4 , x \text{ real}\}$.

The graphs of the following two relations are constructed in the same way:

$\{(x, y) \mid y = -1 , x \text{ real}\}$



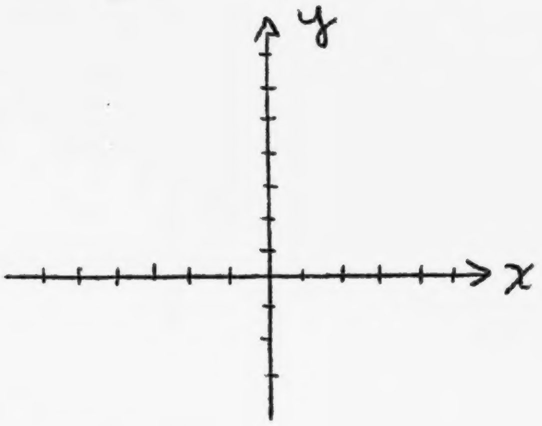
$\{(x, y) \mid y = 3 , x \text{ real}\}$



Construct the graph for the relation

$\{(x, y) \mid x = -1 , x \text{ real} \}$:

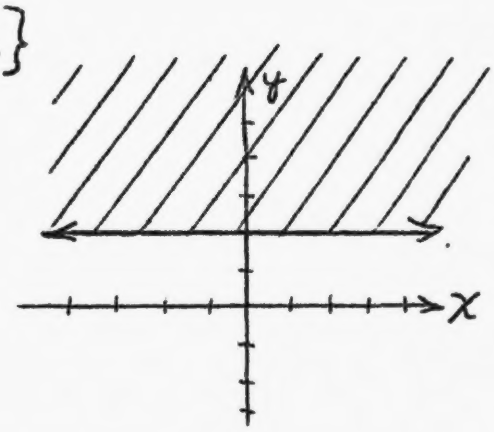
13.



The solution is shown on page 30 .

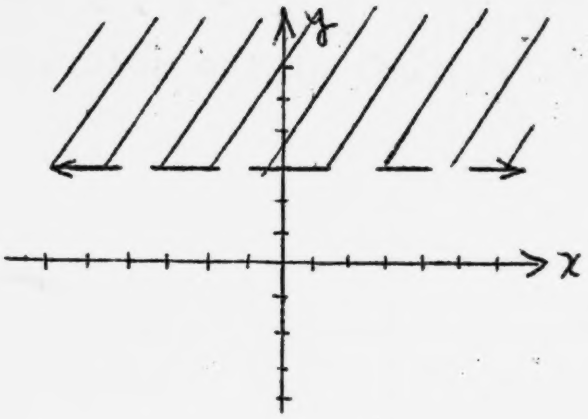
If the relation involves an inequality then the graph includes a shaded region. Several relations and their respective graphs are shown below:

i. $\{(x, y) \mid y \geq 2 , x \text{ real}\}$



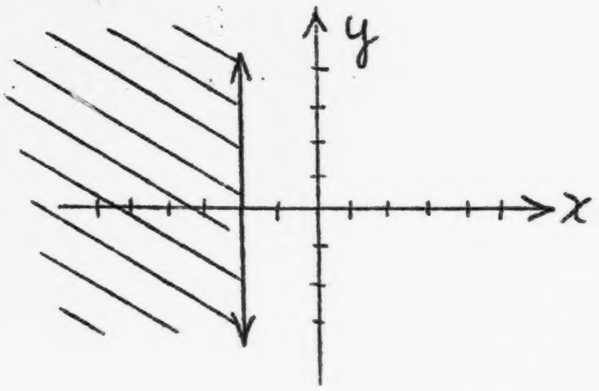
In (i.) notice that the ordered pairs in the solution set consist of all ordered pairs for which the y coordinate is ≥ 2 . Select any point in the shaded region or on the line to convince yourself that its y coordinate is ≥ 2 .

ii. $\{(x, y) \mid y > 3 \text{ , x real}\}$

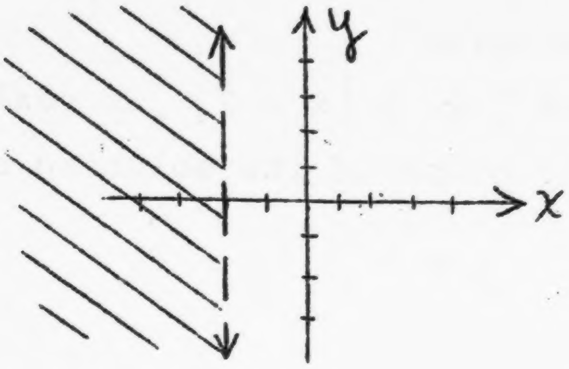


In (ii.), notice that $>$ and the dashed line "go together."

iii. $\{(x, y) \mid x \leq -2 \text{ , x real}\}$



iv. $\{(x, y) \mid x < -2 \text{ , x real}\}$

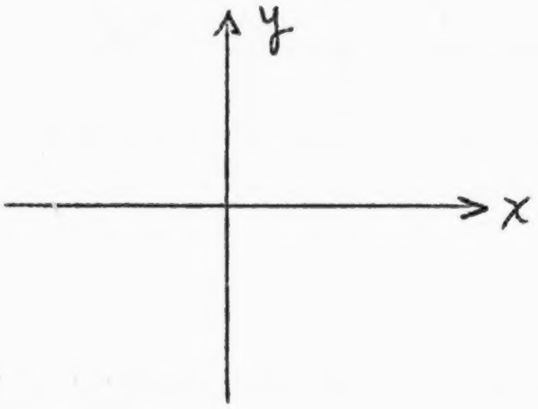


To assure yourself that you can construct graphs when given relations, practice on these problems:

14. Given the relation

$$\left\{ (x, y) \;\middle|\; y < -2 \text{ , } x \text{ real} \right\}$$

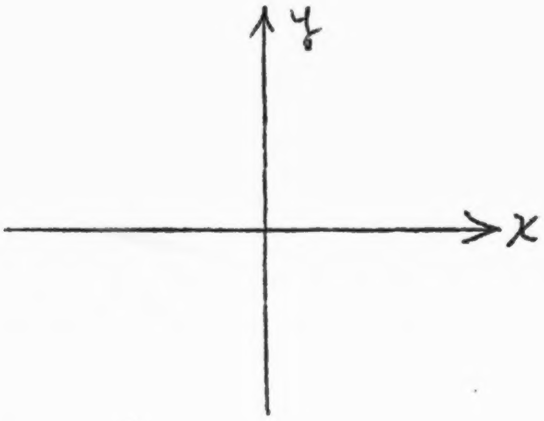
Construct a graph of its solution set.



15. Given the relation

$$\left\{ (x, y) \;\middle|\; y = |x - 3| \text{, } x \text{ real} \right\}$$

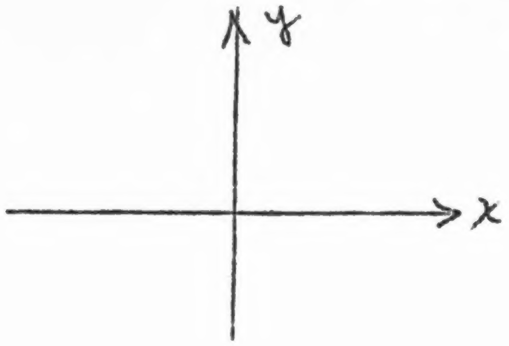
Construct the graph of its solution set.



16. Given the relation

$$\left\{ (x, y) \;\middle|\; y > |x - 2| \text{ , } x \text{ real} \right\}$$

Construct the graph of its solution set.

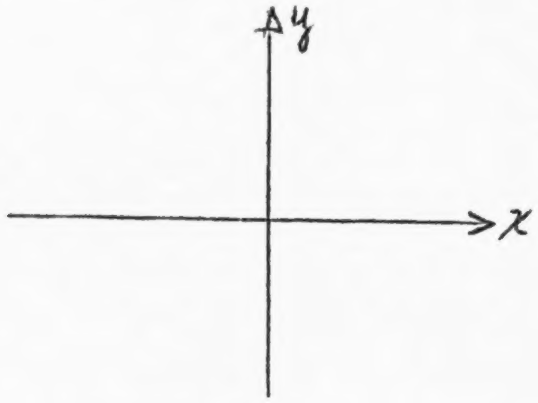


Note: Remember that > and a dashed edge "go together."

17. Given the relation

$$\left\{ (x, y) \mid y \leq 1 - x \text{ , } x \text{ real} \right\}$$

Construct the graph of its solution set.



Note: Remember that ≤ and a solid edge "go together."

Solutions of these problems are shown on page 31 .

Answers to Performance Tasks

Pages 4, 5, and 6.

1a. $y = -11$
1b. $y = -7$
1c. $y = -4$
2a. $y = 8$
2b. $y = 5$
2c. $y = 2$
3a. $y = 4$
3b. $y = 2$
3c. $y = 0$
3d. $y = 2$
3e. $y = 4$

Pages 12 and 13.

(The tables are correct solutions. However, your table may be different if you chose different values of x .)

4.

x	-3	-1	1
y	-6	-4	-2

5.

x	-2	0	2
y	3	5	7

6.

x	-1	1	3
y	4	2	0

7.

x	-3	-1	1
y	1	-1	-3

Page 17.

(These tables are correct solutions. However, your table may be different except the center position depending upon what values of x you selected. There must be five ordered pairs.)

8.

x	-3	-1	1	3	5
y	4	2	0	2	4

9.

x	-6	-4	-2	0	2
y	4	2	0	2	4

10.

x	-1	1	3	5	7
y	4	2	0	2	4

Page 21.

(If you chose different values of x , your table may be different, except for the center position.)

11.

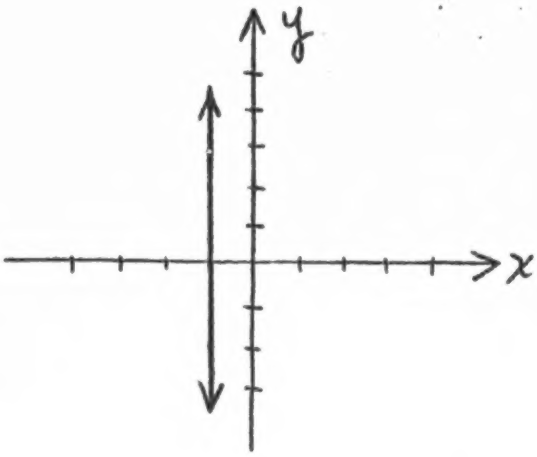
x	-6	-4	-2	0	2
y	4	2	0	2	4

12.

x	0	2	4	6	8
y	4	2	0	2	4

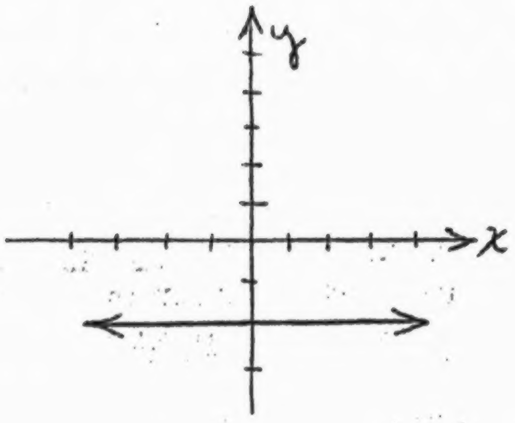
Page 25.

13.



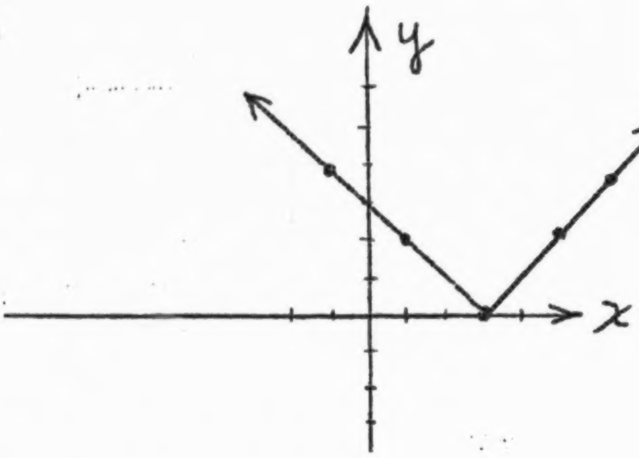
Page 27.

14.



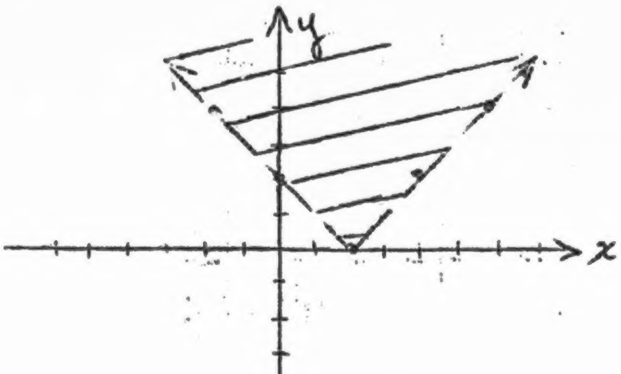
15.

x	-1	1	3	5	7
y	4	2	0	2	4



16.

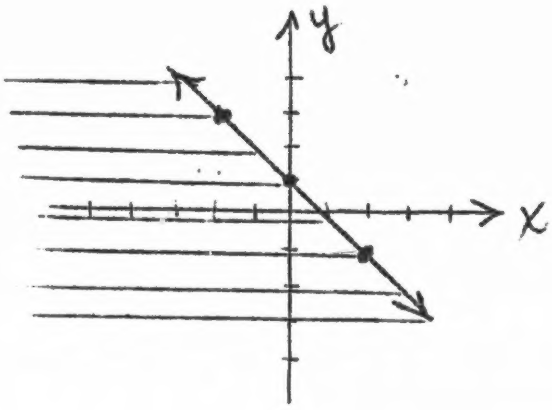
x	-2	0	2	4	6
y	4	2	0	2	4



Page 28.

17.

x	-2	0	2
y	3	1	-1



VIII-a. TWO RELATIONS

In Lesson VII-b you were taught the skill of constructing graphs of relations on a Cartesian plane. In your practice problems you were asked to construct the graph of a single relation on a Cartesian plane. One of the problems and its solution was the following:

GIVEN the relation

$$\left\{(x, y) \mid y > |x - 2|, x \text{ real}\right\}$$

CONSTRUCT the graph of its solution set.

Solution:

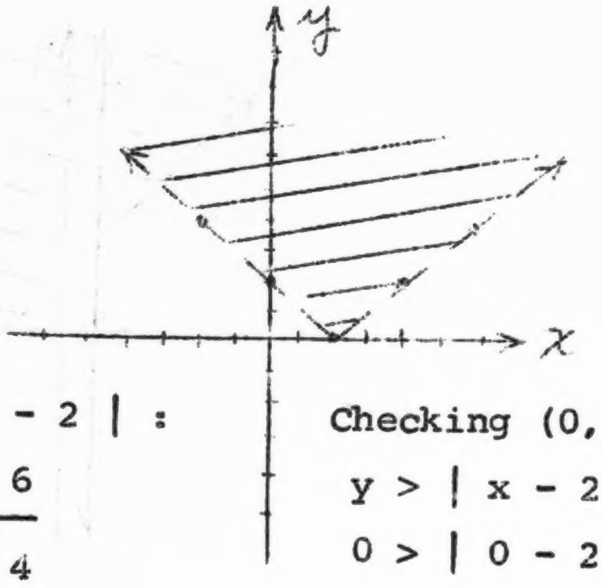


Table for $y = |x - 2|$:

x	-2	0	2	4	6
y	4	2	0	2	4

Checking (0, 0) :

$y > |x - 2|$

$0 > |0 - 2|$

$0 > |-2|$

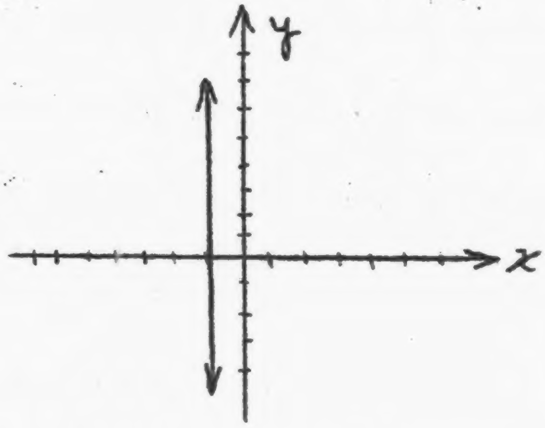
$0 > 2$ is not true

A review of the above procedure is found in Appendix A-3 (white folder). Use Appendix A-3 as a reference for your work in Lesson VIII-a.

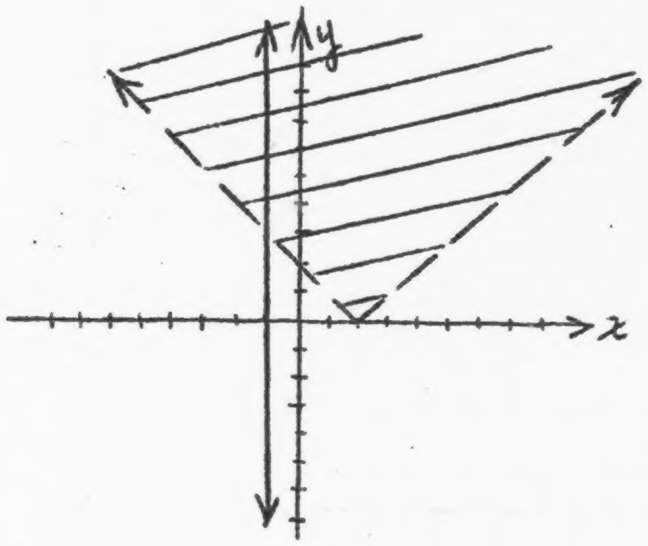
Another problem and its solution in Lesson VII-b was the following:

CONSTRUCT the graph of the solution set for the relation $\{(x, y) \mid x = -1 , x \text{ real}\}$.

Solution:



If the two graphs shown above were constructed on the same Cartesian plane, we would have:



That is, given the two relations

$$\{(x, y) \mid y > |x - 2| , x \text{ real}\}$$

and $$\{(x, y) \mid y = -1 , x \text{ real}\} ,$$

the result of constructing a graph of each of their solution sets on the same Cartesian plane would be the above figure. It is important to note that the graphs are graphs of the solution set of each relation. That is, each graph satisfies the conditions of its relation.

Let's consider another two relations and construct the graphs of each of their solution sets. The task can be stated:

GIVEN the two following relations

$$\{(x, y) \mid y \leq 2 - x\ ,\ x \text{ real}\}$$

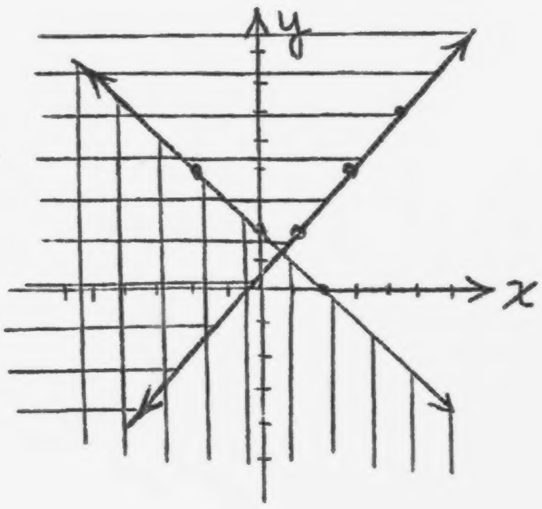
and $$\{(x, y) \mid y \geq x + 1\ ,\ x \text{ real}\}$$

CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.

The procedure for performing the task would be as follows:

For $\{(x, y) \mid y \leq 2 - x\ ,\ x \text{ real}\}$,

x	-2	0	2
y	4	2	0



For $\{(x, y) \mid y \geq x + 1\ ,\ x \text{ real}\}$,

x	1	3	5
y	2	4	6

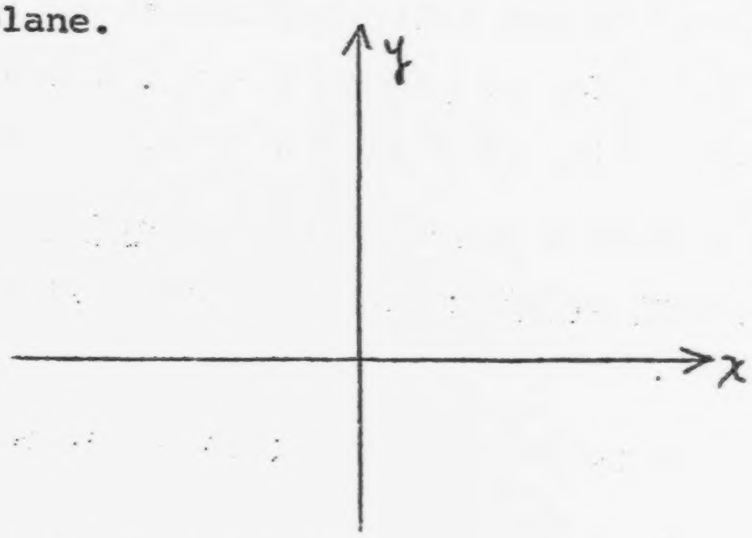
Here are three practice problems for you to work:

1. GIVEN the two relations

$$\{(x, y) \mid y = |x - 1|, \ x \text{ real}\}$$

and $\{(x, y) \mid y > 3 , \ x \text{ real}\}$

CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.

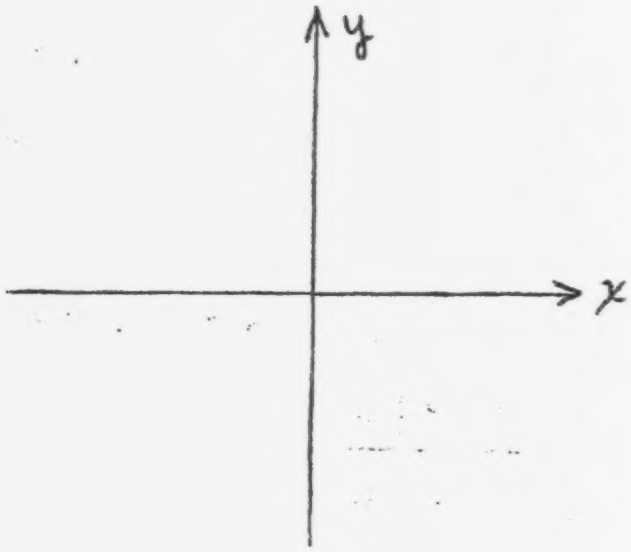


2. GIVEN the two relations

$$\{(x, y) \mid y \geq |x + 2| , \ x \text{ real}\}$$

and $\{(x, y) \mid y \leq 2 , \ x \text{ real}\}$

CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.

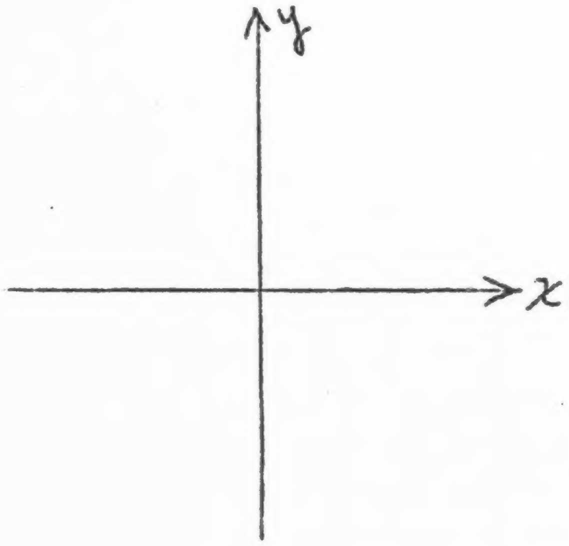


3. GIVEN the two relations

$$\{(x, y) \mid y < x + 2 \text{ , } x \text{ real}\}$$

and $$\{(x, y) \mid y > 3 - x \text{ , } x \text{ real}\}$$

CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.



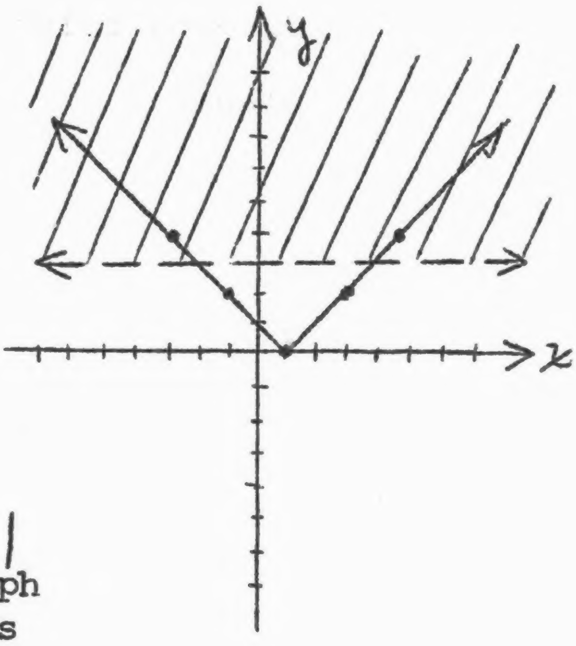
Solutions for these three problems are shown on page 6.

Answers to Performance Tasks

Page 4.

1.

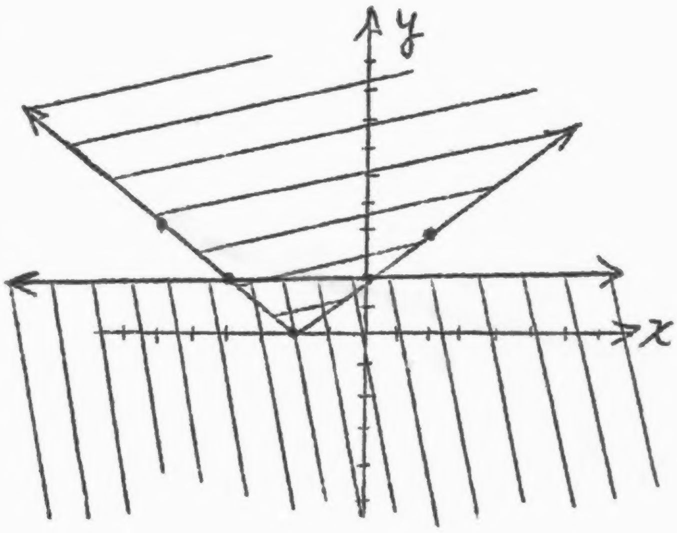
x	-3	-1	1	3	5
y	4	2	0	2	4



(Note: Since $y = |x - 1|$ is an equation, the graph of its solution set does not have a shaded region.)

2.

x	-6	-4	-2	0	2
y	4	2	0	2	4

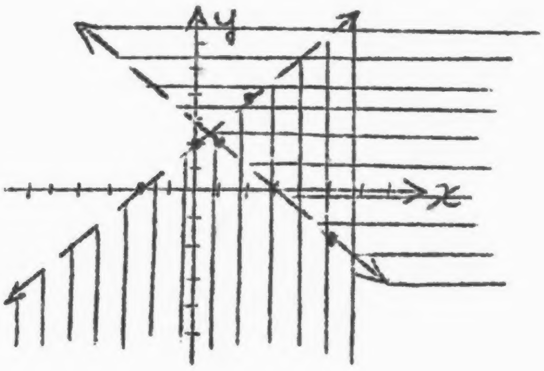


Page 5.

3.

x	-2	0	2
y	0	2	4

x	1	3	5
y	2	0	-2



VIII-b. SYSTEMS

Let us consider two relations *at the same time* (i.e., simultaneously) and ask what set of ordered pairs satisfy the conditions of both relations. Suppose we ask this question of these two relations:

$$\left\{(x, y) \mid y > |x - 2| \ , \ x \text{ real}\right\}$$
$$\text{and } \left\{(x, y) \mid x = 4 \ , \ x \text{ real}\right\}$$

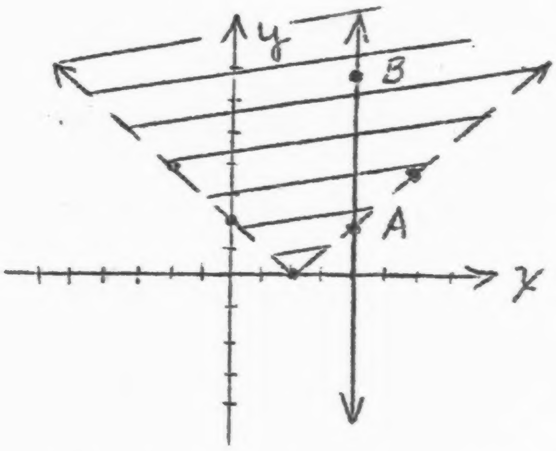
Since we are asking for the set of ordered pairs *that satisfy the conditions of both relations*, we are really seeking the *$\cap$ set* of these two relations.

Hence, we are asking:

$$\left\{(x, y) \mid y > |x - 2| \ , \ x \text{ real}\right\} \cap \left\{(x, y) \mid x = 4, \ x \text{ real}\right\} =$$

We can determine the $\cap$ set by constructing the graph of the solution set for each relation on the same Cartesian plane:

x	-2	0	2	4	6
y	4	2	0	2	4



The set of points (or set of ordered pairs) which satisfy the conditions of both graphs is the $\cap$ set consisting of half-line AB.

A more convenient way of designating the $\cap$ of two relations which are considered simultaneously is

$$\left\{\begin{array}{l}\{(x, y) \quad y \qquad x - 2 \ , \ x \text{ real}\} \\ \{(x, y) \quad x \ = \ 4 \ , \ x \text{ real}\}\end{array}\right.$$

When a set of two or more relations are considered simultaneously, we call the set a *system of relations*. Hence, we have here a system of two relations. The $\cap$ set is the set of points which *satisfies the conditions of both relations at the same time*. Thus, the $\cap$ set of the two relations is called the *solution set of the system*.

Definition 10: The *solution set* of a system of two relations is the $\cap$ set of the two relations.

The solution set of the system

$$\left\{\begin{array}{l}\{(x, y) \mid y > |x - 2| \ , \ x \text{ real}\} \\ \{(x, y) \mid x = 4 \ , \ x \text{ real}\}\end{array}\right.$$

is half-line AB in the above figure. When we speak of constructing a *graph of the solution set of a system of relations*, we mean the graph of the two relations with the $\cap$ set denoted by lettered points. The naming of the solution set is accomplished by merely naming the $\cap$ set utilizing the lettered points.

Suppose we were asked to perform the following task:

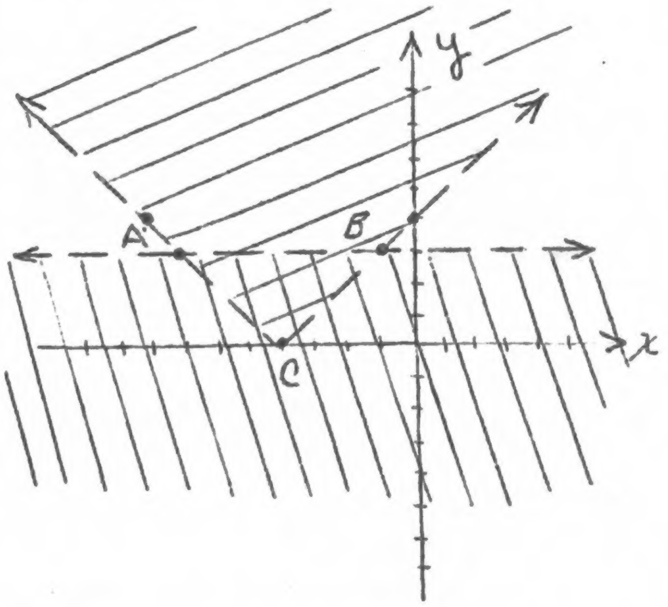
GIVEN the following system of two relations

$$\begin{cases}\{(x, y) \mid y > |x + 4| , \ x \text{ real}\} \\ \{(x, y) \mid y < 3 , \ x \text{ real}\}\end{cases}$$

CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

The solution of this problem would be the following:

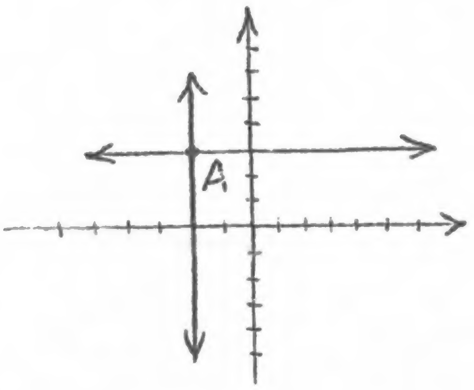
x	-8	-6	-4	-2	0
y	4	2	0	2	4



Interior of $\triangle$ ABC

For each of the following three systems, the graph of each system has been constructed and the solution set of each system has been named:

i. $$\begin{cases}\{(x, y) \mid y = 3 , \ x \text{ real}\} \\ \{(x, y) \mid x = -2 , \ x \text{ real}\}\end{cases}$$

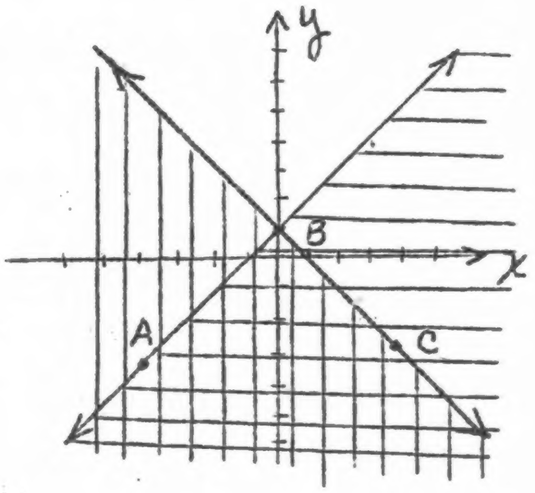


Point A

ii. $\left\{ \begin{array}{l} \{(x, y) \mid y \le x + 1 , x \text{ real}\} \\ \{(x, y) \mid y \le 1 - x , x \text{ real}\} \end{array} \right.$

x	-1	1	3
y	0	2	4

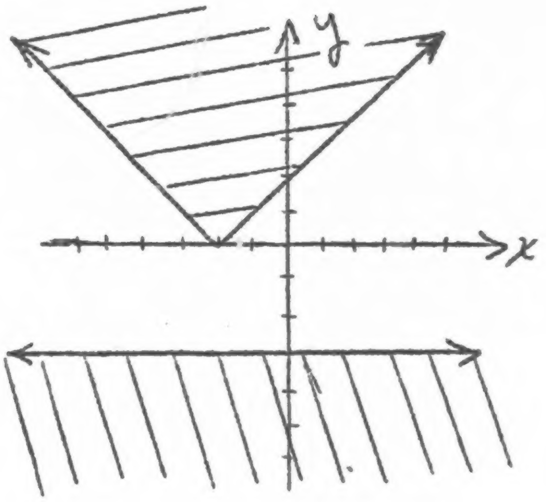
x	-1	1	3
y	2	0	-2



$\angle ABC \cup$ (Interior of $\angle ABC$)

iii. $\left\{ \begin{array}{l} \{(x, y) \mid y \ge |x + 2| , x \text{ real}\} \\ \{(x, y) \quad y \le -3 , x \text{ real}\} \end{array} \right.$

x	-6	-4	-2	0	2
y	4	2	0	2	4



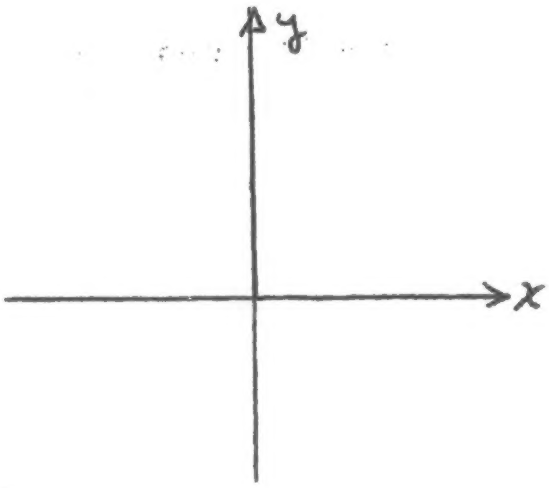
Solution set = ϕ

In (iii.), the solution set is the null set ϕ since there is no $\cap$ set for the two relations.

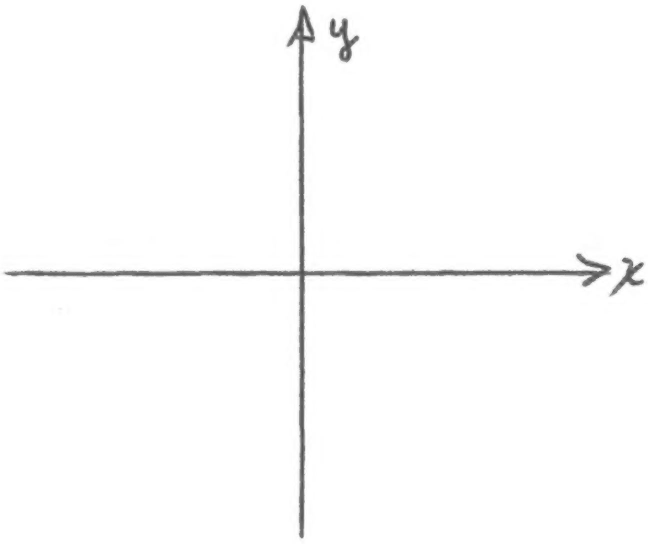
The following three problems are for your practice.

GIVEN the system of two relations in each of the three problems; CONSTRUCT a graph of the solution set of each system and NAME the solution set of each system in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

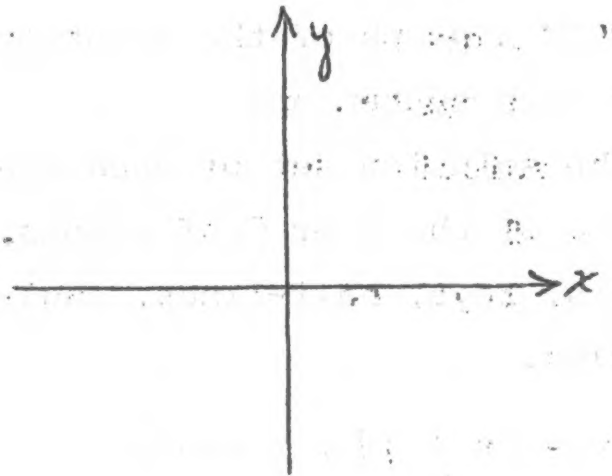
1. $\begin{cases} \{(x, y) \mid y = |x + 3|, \ x \text{ real}\} \\ \{(x, y) \mid y = 4, \ x \text{ real}\} \end{cases}$



2. $\begin{cases} \{(x, y) \mid y \geq -3, \ x \text{ real}\} \\ \{(x, y) \mid y \geq x - 2, \ x \text{ real}\} \end{cases}$



3. $$\left\{\begin{array}{l}\{(x, y) \mid y \le x + 2,\ x \text{ real}\} \\ \{(x, y) \mid x = 1,\ x \text{ real}\}\end{array}\right.$$



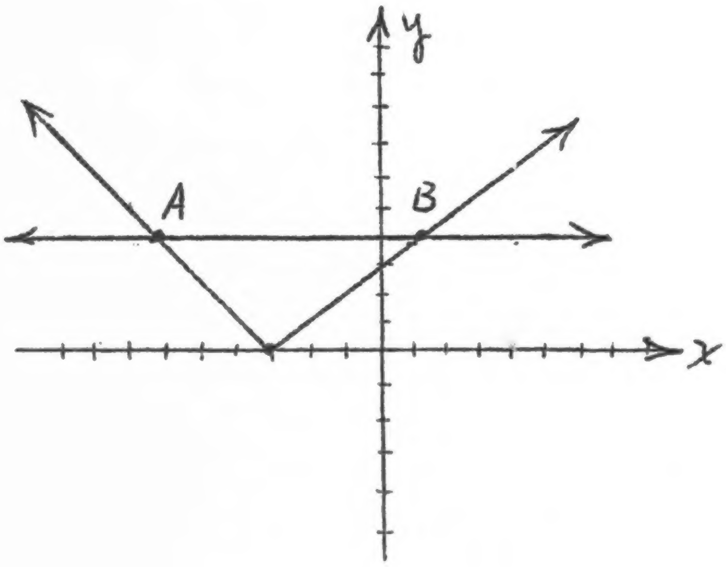
Answers to these problems are shown on page 7.

Answers to Performance Tasks

Page 5

1.

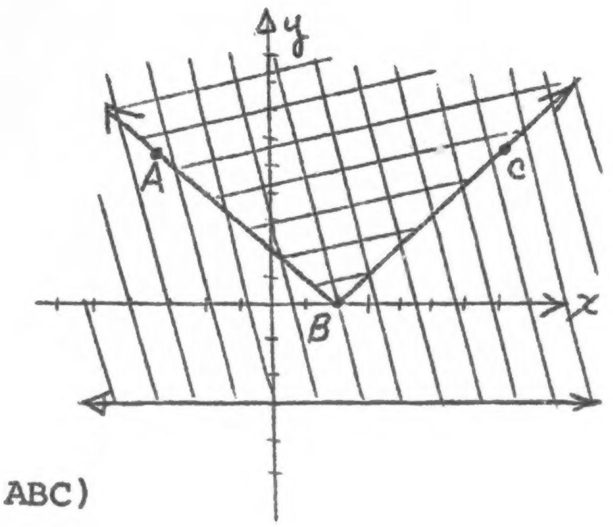
x	-7	-5	-3	-1	1
y	4	2	0	2	4



$A \cup B$

2.

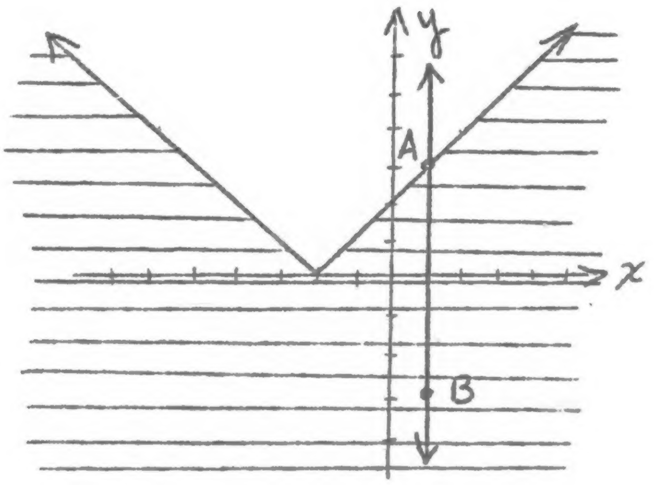
x	-2	0	2	4	6
y	4	2	0	2	4



∠ ABC ∪ (Interior of ∠ ABC)

3.

x	-6	-4	-2	0	2
y	4	2	0	2	4



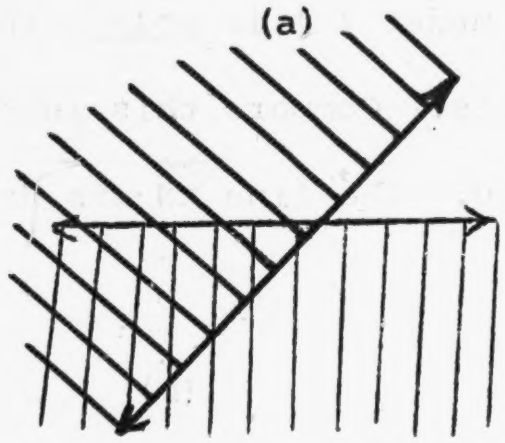
$\overrightarrow{AB}$

APPENDIX A-1

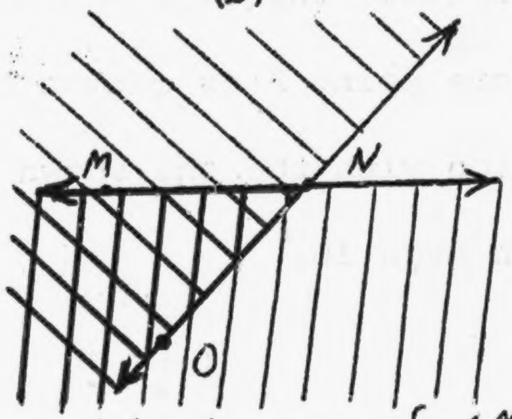
of the intersection is also given. Remember that the intersection of any two graphs consists of the points which are common to both graphs. THE INTERSECTION SET IS NAMED BY ASSIGNING LETTER NAMES TO POINTS AND USING THESE LETTERS TO NAME THE INTERSECTION SET.

i.

(a)



(b)

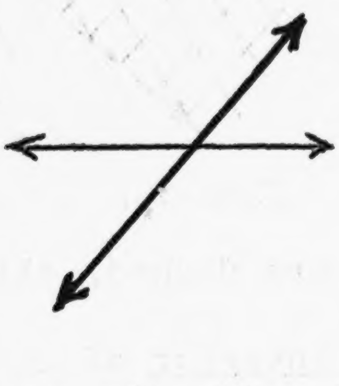


$\angle MNO \cup$ (Interior of $\angle MNO$)

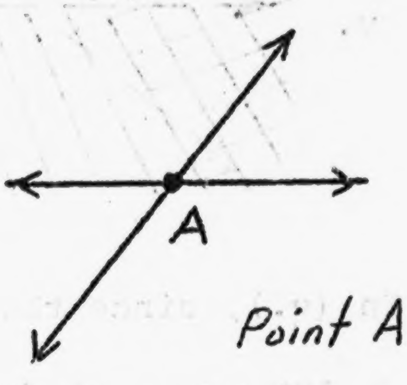
In (i.) the $\angle$ MNO is common to both sets. In addition, the interior of $\angle$ MNO is common to both sets. Hence, the intersection consists of the $\angle$ MNO in union with its interior. Notice that the points which are common can be located by noting where both green and black overlap in (a).

ii.

(a)



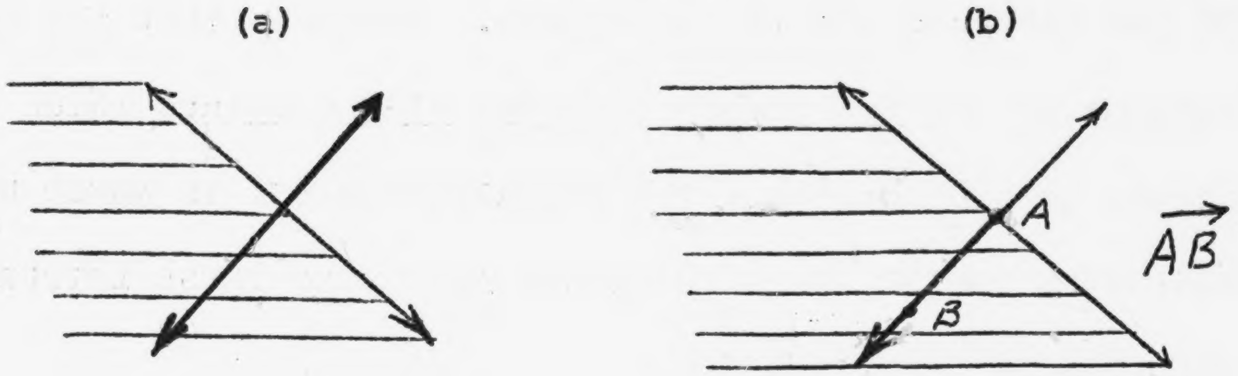
(b)



Point A

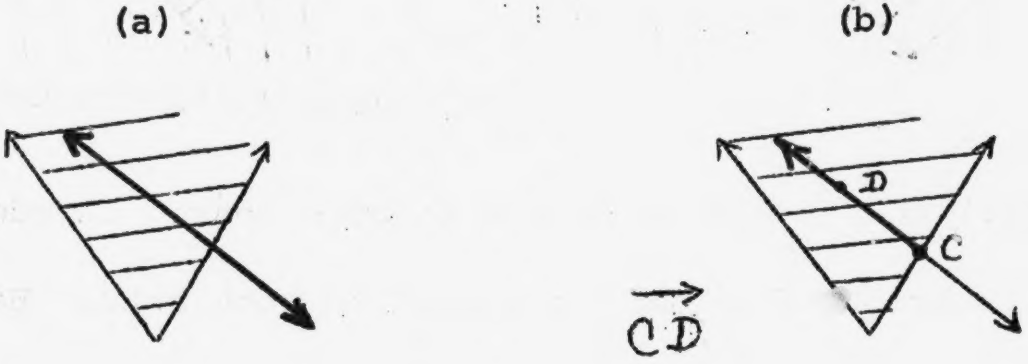
In (ii.) the situation is similar to the figure on page 8.

iii.

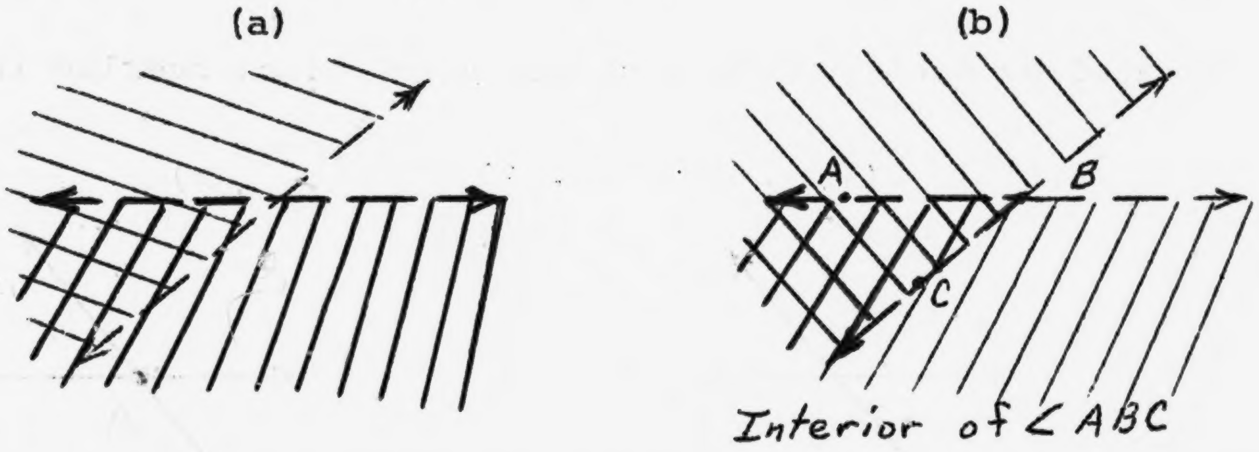

(a) (b)
A
B
$\overrightarrow{AB}$

In (iii) the line shown in black under (a) is solid, therefore point A is common to both sets. Compare this intersection with the one shown on page 10. The line $\overleftrightarrow{MN}$ was dashed on page 10.

iv.


(a) (b)
D
C
$\overrightarrow{CD}$

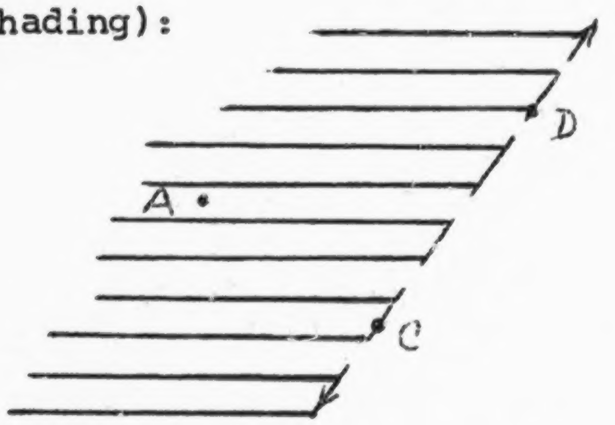
v.


(a) (b)
A
B
C
Interior of ∠ ABC

In (v.), since the sides of ∠ ABC are dashed, all points on ∠ ABC are excluded. However, the interior of ∠ ABC is common to both sets in (a). Hence, the intersection set is the interior of ∠ ABC.

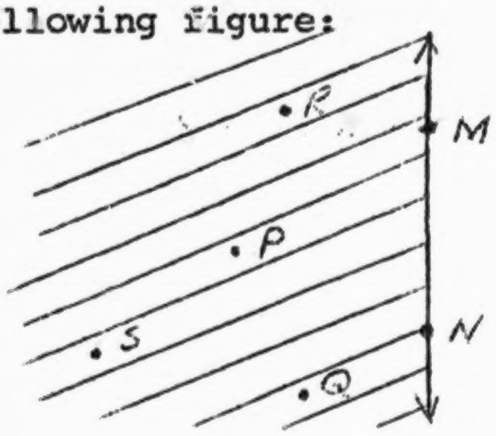
APPENDIX A-2

Half-plane A-CD would be represented as follows (red shading):



You have probably noticed that if you have three points, two of which are on a line while the third point is not on the line, then you have a line and a half-plane represented.

Let's look at this a little more carefully. Suppose we have the following figure:



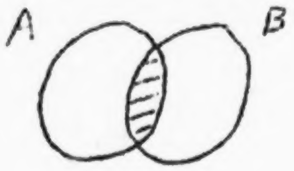
The point P is a point on the half-plane P-MN. The line $\overleftrightarrow{MN}$ is shown solid, therefore the U of the half-plane P-MN and the line $\overleftrightarrow{MN}$ is represented in the figure. The point Q is on the same half-plane that P is on. Thus, we could call the half-plane Q-MN. That is, any point on the half-plane can be used to name the half-plane.

Point R is also on the half-plane. That is, the points P, Q, and R have a property in common: they are in the __________ half-plane. Did you write "same"? That is correct, they are each on the same side of the line $\overleftrightarrow{MN}$.

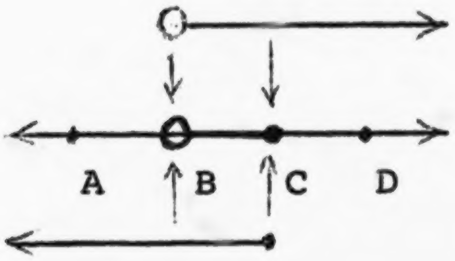
Let me re-phrase this idea. Suppose point R is on the same half-plane as points P, Q, and S. Before going further, let's note that any one of these points can be thought of as being used to specify which half-plane is being referred to. Now, we have given that R is on the same half-plane as P, Q, and S. O.K., then if R is to the left of line $\overleftrightarrow{MN}$, it follows that P, Q, and S are to the left of line MN. In fact, IF ONE POINT ON THE HALF-PLANE IS TO THE LEFT OF LINE MN, THEN ALL THE POINTS OF THE HALF-PLANE ARE TO THE LEFT OF LINE MN.

Sounds foolish? You're probably saying it's too obvious to spend so much time with. Well, we'll leave it for now and look at something more interesting.

We've looked at the intersection of Venn diagrams:



and of rays, line segments, etc.:



APPENDIX A-3

When a relation involves an inequality, we use the table of ordered pairs to graph the edge of the region which is to be shaded. Consider the relation $\{(x, y) \mid y \geq |x + 3|, x \text{ real}\}$. In order to construct the graph of this relation we perform the following three steps:

i. Construct the table of ordered pairs of the equation $y = |x + 3|$.

ii. Plot the ordered pairs from the table to locate the edge of the shaded region of the graph of the inequality.

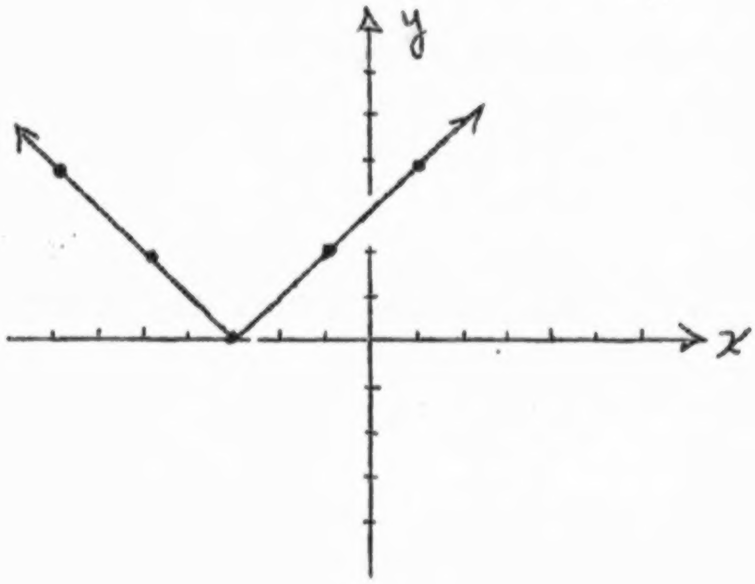
iii. Determine as we did in Lesson VII-a whether or not the origin is in the shaded region; shade the appropriate region.

Let's go through each of these steps for the relation $\{(x, y) \mid y \geq |x + 3|, x \text{ real}\}$:

i.

x	-7	-5	-3	-1	1
y	4	2	0	2	4

ii.



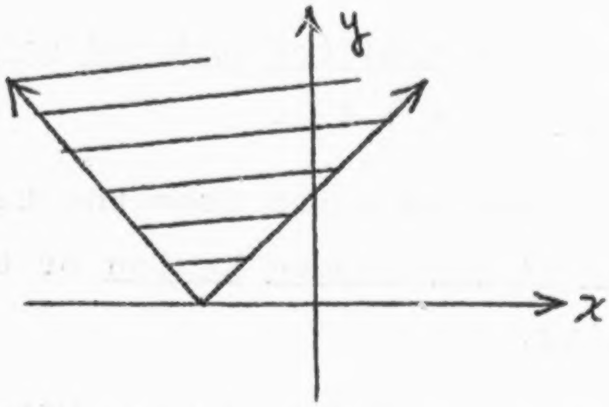
iii. $y \geq |x + 3|$

substituting (0, 0) , $0 \geq |0 + 3|$

$0 \geq |+3|$

$0 \geq 3$ is not true .

Therefore, we have



If you were asked to construct the table of ordered pairs from which the graph of the relation

$\left\{ (x, y) \,\middle|\, y < |x - 1| , x \text{ real} \right\}$ could be constructed,

you would perform step (i.) as follows:

Set: $y = |x - 1|$

Choose the values of x :

x	-3	-1	1	3	5
y			0		

Complete the construction of the table:

x	-3	-1	1	3	5
y	4	2	0	2	4

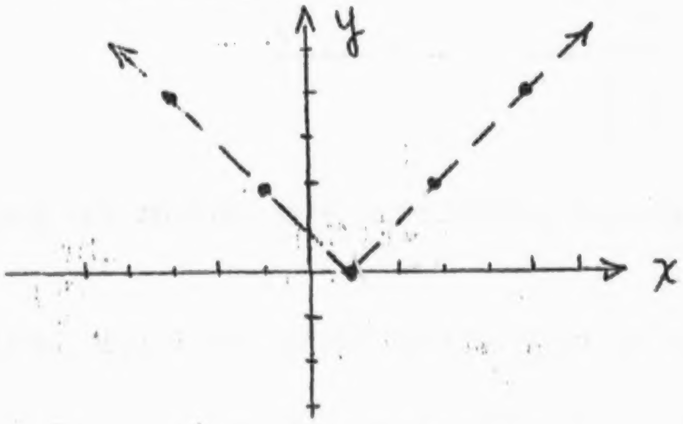
Notice step (i.) is the same whether the inequality is of the form ≤ , ≥ , < , or > .

Later in this lesson you will be asked to construct a graph for a relation similar to $\{(x, y) \mid y < |x - 1|,$ $x \text{ real}\}$. Steps (i.), (ii.), and (iii.) should be followed:

i. We have already constructed the table.

x	-3	-1	1	3	5
y	4	2	0	2	4

ii. Since we have $<$, the edge of the shaded region will be dashed:



iii. Checking the origin (0, 0)

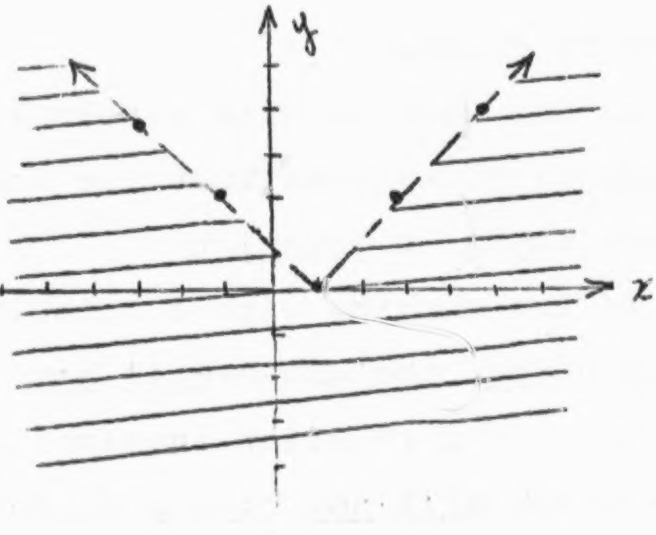
$$y < |x - 1|$$

substituting (0, 0), $0 < |0 - 1|$

$$0 < |-1|$$

$0 < 1$ is true

Hence, we have



Construct the table of ordered pairs from which the graph of each of the two following relations could be constructed:

11. $\left\{(x, y) \mid y \geq |x + 2|, x \text{ real}\right\}$

x	
y	

12. $\left\{(x, y) \mid y > |x - 4|, x \text{ real}\right\}$

x	
y	

Solutions for these problems are shown on page 30 .

Now, let's try bring what we have learned about constructing graphs of relations into a clear focus. We know that:

i. If we are working with an inequality with $\leq$ or $\geq$, the edge of the shaded region on the plane will be solid;

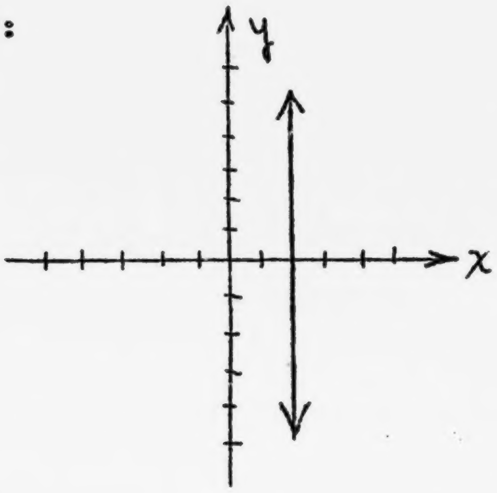
ii. If we are working with an inequality with $<$ or $>$, the edge of the shaded region on the plane will be dashed;

iii. If we are working with a relation involving an inequality, we construct the table of ordered pairs using an equation;

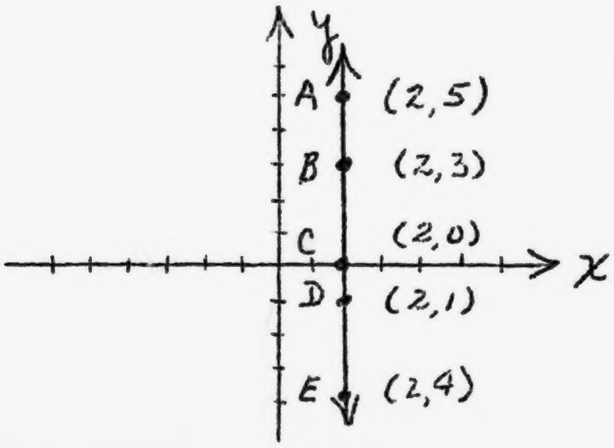
iv. If we are working with a relation involving an inequality, the graph will have a shaded region, while if the relation involves only an equation, the graph will not have a shaded region;

v. If we are working with a relation involving an inequality, the location of the shaded region is determined by whether or not the origin (0, 0) satisfies the conditions of the inequality.

There are some relations for which it is not necessary to construct tables of ordered pairs in order to construct their graphs. Consider the line shown on the Cartesian plane shown below:



Now, let's notice the coordinates of some points on the line.



The value of x for every point is x = 2 . It doesn't seem to matter what the value of y is for points on the line, the x = 2 . Hence, the line is the graph of the relation

$\{(x, y) \mid x = 2 , x \text{ real}\}$.

APPENDIX B

INTRODUCTION, BEHAVIORAL OBJECTIVES FOR EACH ACTIVITY, AND TEXT MATERIAL INSERTS (TREATMENT T_2)

INTRODUCTION

The materials for your next unit of instruction were written by the University of Maryland Mathematics Project. Perhaps you will find that the format of the materials will differ from the textbooks that you have previously used. Ordinary textbooks usually require homework. Homework is not required for this material. In this material, all your work will be done individually in class. You will be asked to answer questions by writing on the blanks provided throughout the materials or by performing the other tasks requested.

These materials are designed around what you do. If you work on each task at the point it is called for, you will find that the materials have a completeness about them that is not in an ordinary textbook. You will also find that your response at each step will help you to acquire the skills upon which you will be tested. FOR YOUR BENEFIT do not read further until you have completed each task the way you think it should be done. (No one is going to count the number of questions answered correctly--the tasks are for your benefit.) If you answer incorrectly or perform a task incorrectly, re-read the questions and correct your original response.

At the beginning of each class, your instructor will inform you of the maximum number of lessons which you will be permitted to complete that day. Upon completing a lesson, raise your hand and you will be given another lesson. When you have completed all the lessons permitted for that day, raise your hand and you will be given your "check-up questions." Upon completing your "check-up questions," raise your hand and the instructor will pick up your questions. If you complete the check-up questions before the end of the class period, you may work on some reading assignment. Although everyone will be working at his own pace, it is suggested that you take your time.

You are expected to make arrangements with your instructor for any lesson you might miss because of absence.

Here are some examples to show you how the format will look.

Sets of elements are denoted by the two braces $\{$ and $\}$. Given the two sets of integers

$$\{1, 2, 3, 4, 5\} \quad \text{and} \quad \{4, 6, 7\},$$

circle in red that integer which is common to both sets. Do not read further until you have performed the task!

Did you circle the 5? You didn't? You are correct, the 5 is not in both sets. However, the 4 is in both sets and you are correct if you circled the 4 in both sets.

Given the following two sets

$$\{1, 2, 3\} \quad \text{and} \quad \{4, 5\} ,$$

list all the integers which are in either of the two sets: __________. Have you listed the integers? Do it now before they disappear--or before you read further.

Did you list only the integers 1, 2, and 3? If you did, re-read the question. You were asked to list the integers in either of the two sets. The integers which are in either of the two sets are 1, 2, 3, 4, and 5.

Turn the page. You are ready to begin.

At the start of each lesson you will receive the following:

1. A statement of the objective for the lesson,

and 2. An example of the objective, i.e., of the type of task you will be expected to perform at the end of the lesson.

You are probably more familiar with the words *goals* or *aims* than with *objectives*. The objective of each lesson is *for your benefit*. The objective of a lesson is a description of what a *successful learner will be able to do when he has completed the lesson*. YOU WILL KNOW IN ADVANCE WHAT IS CONSIDERED IMPORTANT BY THE AUTHOR! Advance awareness of the objective will enable you to direct your efforts during the lesson toward learning how to perform the specific task stated in the objective. You will know what you must be able to do *in order to succeed*. With the objective of a particular lesson in view, you will be able to place your emphasis where you know it needs to be placed. *Take time at the start of each lesson* to read carefully the statement and the example of the lesson's objective. If you take the time at the start of each lesson, you will *save yourself a lot of work later*. You will be checked at the end of each lesson on your awareness of the lesson's objective.

There is no statement of objective for Lesson I. Lesson I is simply provided to help you to remember some concepts about sets. There is a definite reason that the statement of the objective was not provided. By not stating an objective for Lesson I, the author is telling you something very important. You will not be expected to remember anything specifically from Lesson I.

Turn the page.

Activity I text material insert.

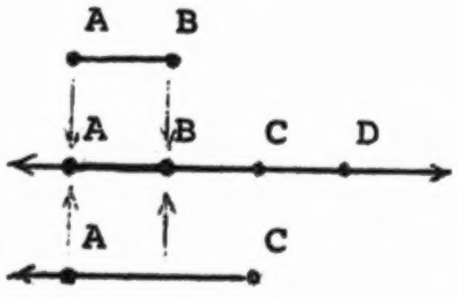
What Do I Have to Learn in This Lesson?

At the end of Lesson II you should be able to perform the following task:

> **GIVEN** the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays;
>
> **NAME** the $\cap$ set in terms of the points, line segments, half-lines, or rays.

Example of the type of task:

> GIVEN line $\overleftrightarrow{AC}$. The set of points $\overline{AB}$ on line $\overleftrightarrow{AC}$ is represented above $\overleftrightarrow{AC}$ and the set of points $\overrightarrow{CA}$ on line $\overleftrightarrow{AC}$ is represented below $\overleftrightarrow{AC}$. The $\cap$ of the two sets of points is shown in red on line $\overleftrightarrow{AC}$.



> NAME the $\cap$ set (shown in red) of the two sets of points $\overline{AB}$ and $\overrightarrow{CA}$.
>
> Answer: $\overline{AB}$

You now are ahead of the game! You know the type of problem the author considers important. Obviously, this is the type of problem you might be tested upon. Pay particular attention when he discusses how to name the $\cap$ set of lines, rays, etc.

Turn the page.

Activity II text material insert.

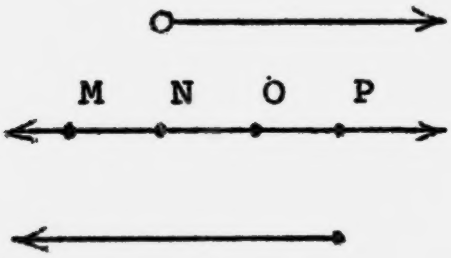
Watch Out For the Objective of Lesson III!

The goal or the objective of Lesson III is to teach you to perform the following task:

> GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays;
> IDENTIFY their $\cap$ by shading on the line.

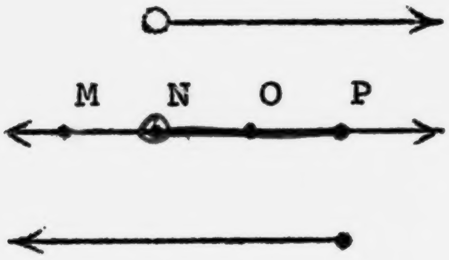
Example of the type of task:

> GIVEN line $\overleftrightarrow{MP}$. One set of points on line $\overleftrightarrow{MP}$ is represented above $\overleftrightarrow{MP}$ while a second set of points on $\overleftrightarrow{MP}$ is represented below $\overleftrightarrow{MP}$.



> IDENTIFY the $\cap$ of the two sets of points by shading line $\overleftrightarrow{MP}$ in red over the appropriate portion.

Solution:



What should you watch for in this lesson? Whenever the author discusses the above topic, perk up! You will be responsible for it "later." When the topic centers around identifying the $\cap$ set of two sets of points such as half-lines, line segments, etc., that is your que to pay particular attention.

Turn the page.

Activity III text material insert.

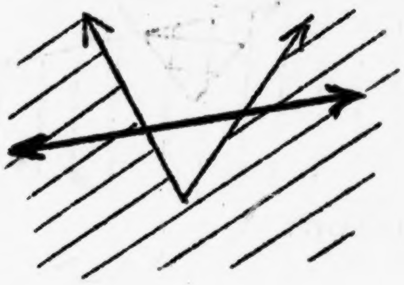
The Objectives of Lesson IV

When you complete Lesson IV, you should be able to perform the following two tasks:

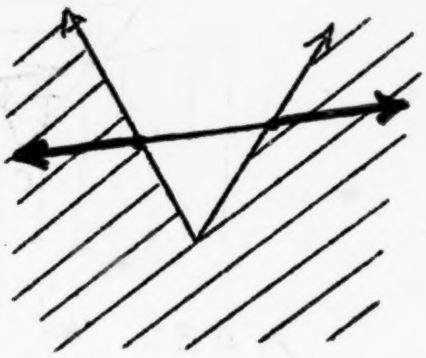
Task 1. GIVEN two graphs plotted on
the same plane;
IDENTIFY, by shading, the ∩
of the two graphs.

Example of Task 1:

GIVEN the following two graphs
(one in green and one in black);
IDENTIFY (by coloring in red)
the ∩ of the two graphs.



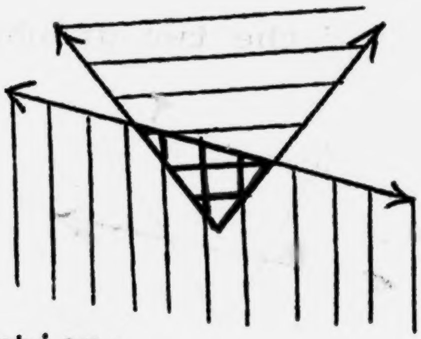
Solution:



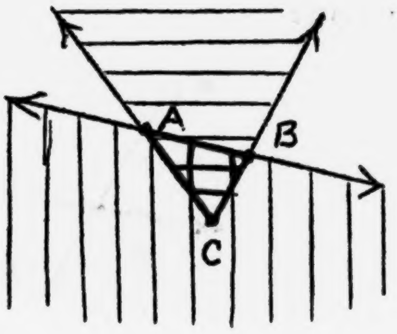
Task 2. GIVEN a figure showing the intersection of two graphs; NAME the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Example of Task 2:

GIVEN the $\cap$ set of the following two graphs; NAME the $\cap$ set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.



Solution:



$\triangle ABC \cup$ (Interior of $\triangle ABC$)

Activity IV text material insert.

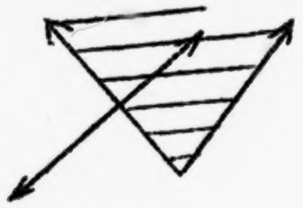
What is Important in Lesson V?

The author has written Lesson V to teach you to perform the following task:

> GIVEN a figure showing a system of two graphs plotted on the same plane; NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

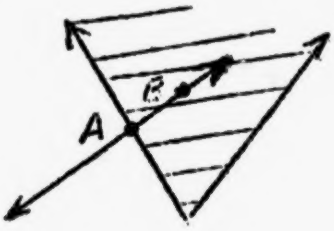
Example of the task:

> GIVEN the following figure of a system of two graphs



> NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:



$\overrightarrow{AB}$

The author considers the above task sufficiently different from the objectives of Lesson IV to require a completely new lesson to teach it. Notice the new words (system and solution set) in the statement of the objective; these are key to this lesson.

Turn the page.

Activity V text material insert.

Lesson VI's Objective

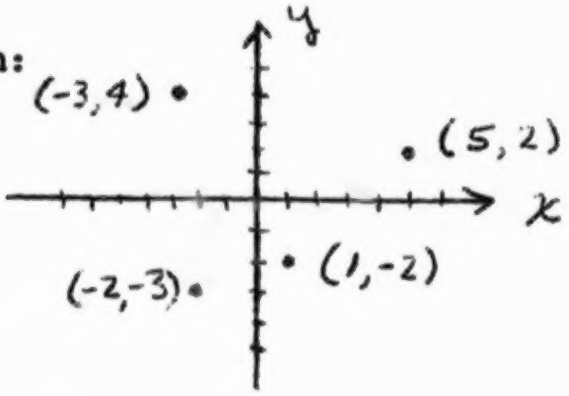
At the end of Lesson VI you should be able to perform the following task:

> GIVEN a set of at least four ordered pairs;
> IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

Example of the task:

> GIVEN the set of ordered pairs:
>
> $\{(-2, -3), (1, -2), (5, 2), (-3, 4)\}$
>
> IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting and naming them by the corresponding ordered pairs.

Solution:



Activity VI text material insert.

The Objectives of Lesson VII

Lesson VII is divided into two parts: Lesson VII-a and Lesson VII-b.

When you complete Lesson VII-a, you should be able to perform the following task:

> GIVEN the graphs of relations from Set A (see below) with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

The Set A referred to in the above statement of the objective of Lesson VII-a is:

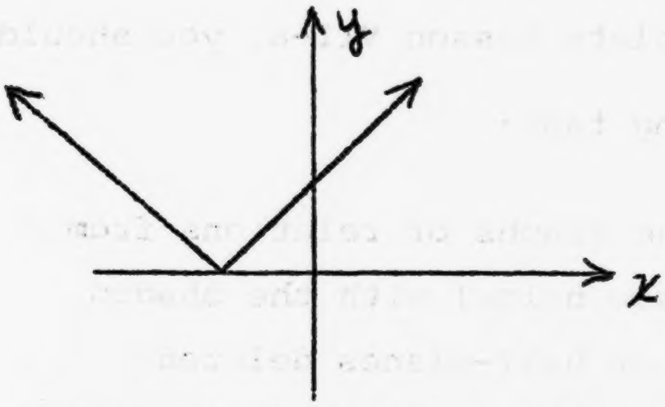
$$\{(x, y) \mid y \geq |x + a|, x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b, x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x, x \text{ real}\}$$
$$\{(x, y) \mid y \geq d, x \text{ real}\}$$
$$\{(x, y) \mid x \geq e, x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$,
or $\leq$;
a, b, c, d, and e are integers.

Example of the task:

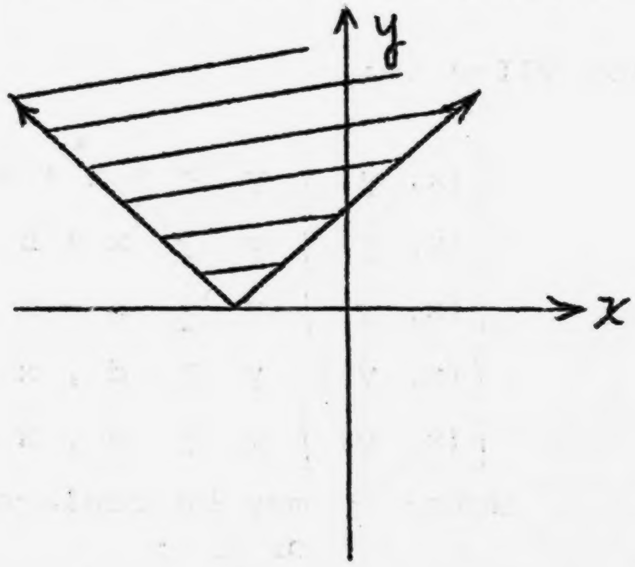
GIVEN the following curve for the relation

$\{(x, y) \mid y \geq |x + 2| \ , x \text{ real}\}$



IDENTIFY, by shading, the region which satisfies the inequality for the relation.

Solution:



Turn the page.

Activity VIIa **text material insert.**

The Objectives of Lesson VII Continued

When you have completed Lesson VII-b, you should be able to perform the following three tasks:

Task 1: GIVEN an equation from
Set B and a value of x,
COMPUTE the corresponding
value of y.

The Set B referred to in the above statement of Task 1 is:

$y = |x + a|$

$y = x + b$

$y = c - x$

Note: a, b, and c are integers

Example of Task 1:

GIVEN the equation

$y = |x - 5|$

with $y = -4$

COMPUTE the corresponding
value of y.

Solution: $y = |-4 - 5| = |-9| = 9$

Task 2: GIVEN a relation from the set of relations shown below, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

The set of relations referred to in the above statement of Task 2 is:

$$\{(x, y) \mid y \geq |x + a|, x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b, x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x, x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.

Example of Task 2:

GIVEN the relation
$$\{(x, y) \mid y \leq 3 - x, x \text{ real}\}$$
CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

Solution:

x	-1	1	3
y	4	2	0

Task 3: GIVEN a relation from
Set A,
CONSTRUCT a graph of
its solution set.

The Set A referred to in the above statement of Task 3 is:

$$\{(x, y) \mid y \geq |x + a| , x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b , x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x , x \text{ real}\}$$
$$\{(x, y) \mid y \geq d , x \text{ real}\}$$
$$\{(x, y) \mid x \geq e , x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$;
a, b, c, d, and e are integers.

Example of Task 3:

GIVEN the relation
$$\{(x, y) \mid y \geq |x + 2| , x \text{ real}\}$$
CONSTRUCT a graph of its solution set.

Solution:

x	-6	-4	-2	0	2
y	4	2	0	2	4

Activity VIIb text material insert.

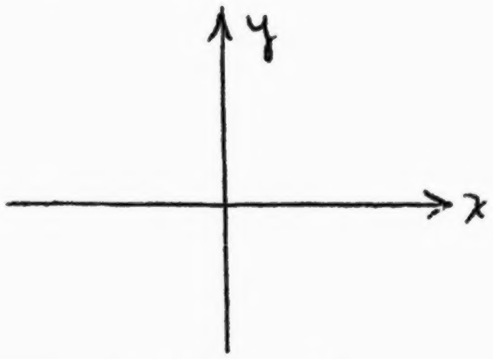
The Objectives of Lesson VIII

Lesson VIII has two parts: VIII-a and VIII-b. The objective of Lesson VIII-a is given here while the objective of Lesson VIII-b is stated later. The objective of Lesson VIII-a is to teach you to perform the following task:

> GIVEN two relations of the type in Set A;
> CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.

Example of the task:

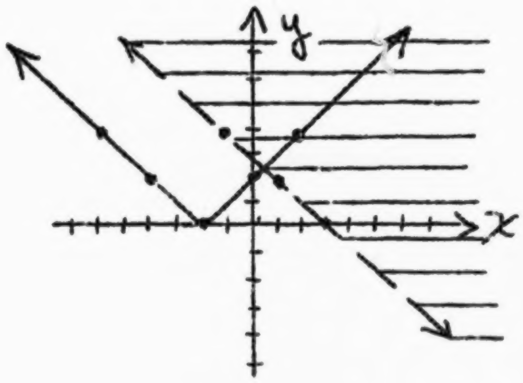
> GIVEN the following two relations
> $$\{(x, y) \mid y > 3 - x\ ,\ x \text{ real}\}$$
> and $$\{(x, y) \mid y = |x + 2|\ ,\ x \text{ real}\}$$
> CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



Solution:

x	-1	1	3
y	4	2	0

x	-6	-4	-2	0	2
y	4	2	0	2	4



Activity VIIIa text material insert.

The Objective of Lesson VIII Continued

When you have completed Lesson VIII-b, you should be able to perform the following task:

> GIVEN a system of two relations
> of the type in Set A;
> CONSTRUCT a graph of its solution set and
> NAME the solution set in terms of the
> $\cup$ or $\cap$ of points, line segments, rays,
> half-lines, angles, or triangles.

The Set A referred to in the above statement of the terminal task is:

$$\{(x, y) \mid y \geq |x + a| , x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b , x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x , x \text{ real}\}$$
$$\{(x, y) \mid y \geq d , x \text{ real}\}$$
$$\{(x, y) \mid x \geq e , x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$,
or $\leq$;
a, b, c, d, and e are integers.

Example of the terminal task:

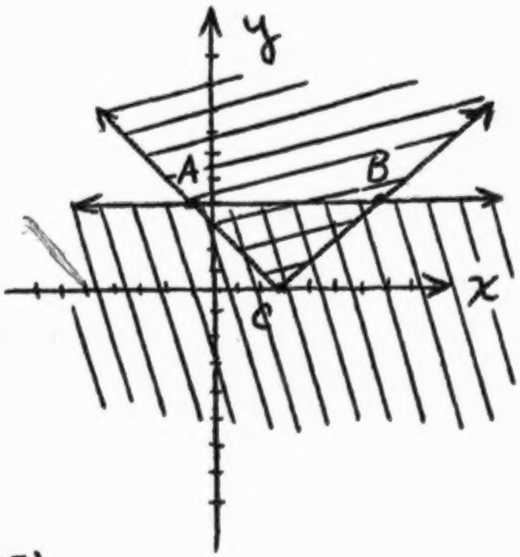
GIVEN the following system of two relations

$$\begin{cases} \{(x, y) \mid y \geq |x - 3|, x \text{ real}\} \\ \{(x, y) \mid y \leq 4, x \text{ real}\} \end{cases}$$

CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cap$ or $\cup$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

x	-1	1	3	5	7
y	4	2	0	2	4



$\triangle$ ABC $\cup$ (Interior of $\triangle$ ABC)

Turn the page.

Activity VIIIb text material insert.

APPENDIX C

INTRODUCTION, LEARNING HIERARCHY INFORMATION AT THE BEGINNING AND END OF THE UNIT OF INSTRUCTION, AND TEXT MATERIAL INSERTS (TREATMENT T_3)

INTRODUCTION

The materials for your next unit of instruction were written by the University of Maryland Mathematics Project. Perhaps you will find that the format of the materials will differ from the textbooks that you have previously used. Ordinary textbooks usually require homework. Homework is not required for this material. In this material, all your work will be done individually in class. You will be asked to answer questions by writing on the blanks provided throughout the materials or by performing the other tasks requested.

These materials are designed around what you do. If you work on each task at the point it is called for, you will find that the materials have a completeness about them that is not in an ordinary textbook. You will also find that your response at each step will help you to acquire the skills upon which you will be tested. FOR YOUR BENEFIT do not read further until you have completed each task the way you think it should be done. (No one is going to count the number of questions answered correctly--the tasks are for your benefit.) If you answer incorrectly or perform a task incorrectly, re-read the questions and correct your original response.

At the beginning of each class, your instructor will inform you of the maximum number of lessons which you will be permitted to complete that day. Upon completing a lesson, raise your hand and you will be given another lesson. When you have completed all the lessons permitted for that day, raise your hand and you will be given your "check-up questions." Upon completing your "check-up questions," raise your hand and the instructor will pick up your questions. If you complete the check-up questions before the end of the class period, you may work on some reading assignment. Although everyone will be working at his own pace, it is suggested that you take your time.

You are expected to make arrangements with your instructor for any lesson you might miss because of absence.

Here are some examples to show you how the format will look.

Sets of elements are denoted by the two braces $\{$ and $\}$. Given the two sets of integers

$$\{1, 2, 3, 4, 5\} \quad \text{and} \quad \{4, 6, 7\},$$

circle in red that integer which is common to both sets. Do not read further until you have performed the task!

Did you circle the 5? You didn't? You are correct, the 5 is not in both sets. However, the 4 is in both sets and you are correct if you circled the 4 in both sets.

Given the following two sets

$$\{1, 2, 3\} \quad \text{and} \quad \{4, 5\} ,$$

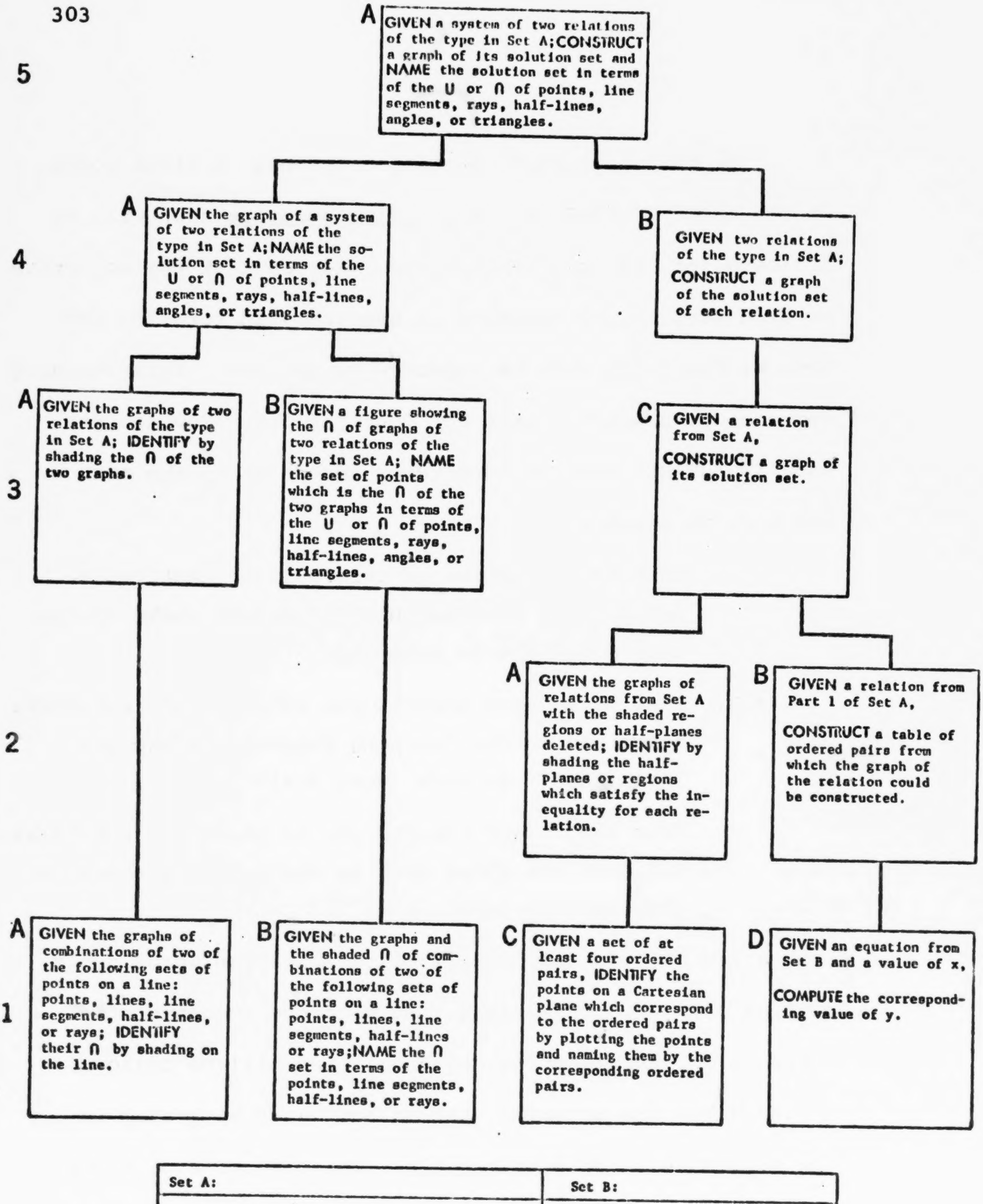
list all the integers which are in either of the two sets: __________. Have you listed the integers? Do it now before they disappear--or before you read further.

Did you list only the integers 1, 2, and 3? If you did, re-read the question. You were asked to list the integers in either of the two sets. The integers which are in either of the two sets are 1, 2, 3, 4, and 5.

Turn the page. You are ready to begin.

LEARNING SEQUENCE

303



5 A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4 A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4 B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3 A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the $\cap$ of the two graphs.

3 B GIVEN a figure showing the $\cap$ of graphs of two relations of the type in Set A; NAME the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

3 C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2 A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

2 B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1 A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their $\cap$ by shading on the line.

1 B GIVEN the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the $\cap$ set in terms of the points, line segments, half-lines, or rays.

1 C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

1 D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:		Set B:
Part 1.	$\{(x,y) \mid y \geq \lvert x + a \rvert,\ x \text{ real}\}$	$y = \lvert x + a \rvert$
	$\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$	$y = x + b$
	$\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$	$y = c - x$
Part 2.	$\{(x,y) \mid y \geq d,\ x \text{ real}\}$	Note: a, b, and c are integers.
	$\{(x,y) \mid x \geq e,\ x \text{ real}\}$	
Note: $\geq$ may be replaced by $=, >, <$, or $\leq$; a, b, c, d, and e are integers.		

We have discussed learning sequences in class today. On the OPPOSITE PAGE is the learning sequence that will be followed for the next three weeks. Notice that the objective of each step in the sequence is expressed in terms of the type of tasks you will be expected to perform. This learning sequence is presented here for your benefit. You now can know IN ADVANCE what is considered important by the author. You know in advance:

1. That the objective of this unit of instruction is that you will be able to perform tasks of the type described in step 5-A.
2. That the author expects you to learn the sub-tasks described in the Learning Sequence before you learn the final task (step 5-A).
3. That the author expects you to learn the sub-tasks and then the final task in the sequence shown on the opposite page.

You are probably more familiar with the words goals or aims than with objectives. As with the sequence of steps (i.e., the learning sequence) that will be followed to teach you the material, the objective of each step in the sequence is for your benefit. The objective of a step is a description of what a successful learner will be able to do when he has completed the lesson for that step.

Awareness of these objectives at the beginning of this unit of instruction will enable you to direct your efforts during the next three weeks toward learning how to perform the specific tasks described by the objectives. You now know what you must do to succeed:

> You are to learn to perform the sub-tasks in the sequence shown so that you will be able to perform the final task.

The author is very much aware that you may not be familiar with the terms used in the statements of objectives in the learning sequence. Hence, FOR YOUR BENEFIT, examples of the type of tasks you're expected to learn to perform at each step are provided in the green folder which you have been given. Take time now to look at the examples and see how they match the statements of the objectives in the Learning Sequence. If you take time now, you will save yourself a lot of work later. Your awareness of the specific objectives will be checked periodically during the next three weeks.

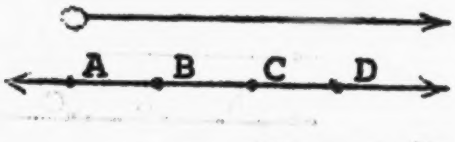
Note that Set A and Set B referred to in several places in the sequence are shown at the bottom of the opposite page.

> Wait! Stop! Have you looked carefully at the Learning Sequence and the examples? Don't rush down a road until you know where it leads.

Turn the page.

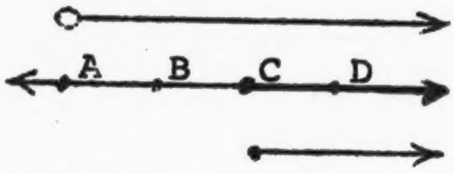
The following examples of the terminal task and the subordinate tasks in the Learning Sequence were given to the students at this point.

GIVEN the line $\overleftrightarrow{AD}$. The graph of one set of points ($\overrightarrow{CD}$) on $\overleftrightarrow{AD}$ is shown below $\overleftrightarrow{AD}$ and the graph of a second set of points (half-line AD) on $\overleftrightarrow{AD}$ is shown above $\overleftrightarrow{AD}$.



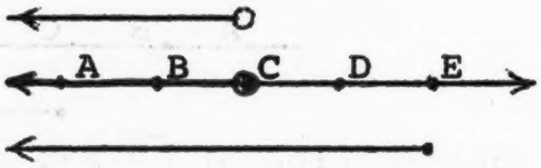
IDENTIFY the intersection of $\overrightarrow{CD}$ and the half-line AD by shading in red on line $\overleftrightarrow{AD}$.

Solution:



Example of 1-A

GIVEN the line $\overleftrightarrow{AE}$. The graph of the set of points half-line CA on $\overleftrightarrow{AE}$ is shown above $\overleftrightarrow{AE}$ and the set of points $\overrightarrow{EA}$ on $\overleftrightarrow{AE}$ is shown below $\overleftrightarrow{AD}$. The $\cap$ of half-line CA and $\overrightarrow{EA}$ is shown in red on $\overleftrightarrow{AE}$.



NAME the $\cap$ set in terms of points, line segments, half-lines, or rays.

Solution: half-line CA

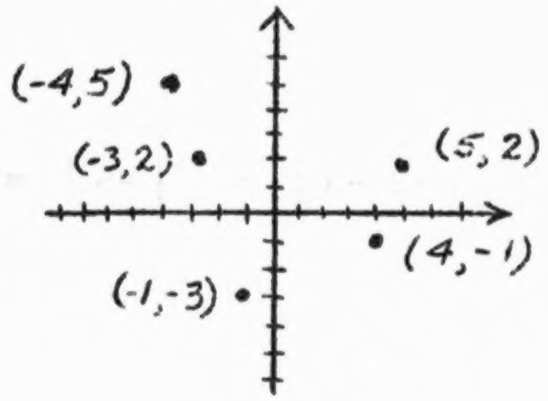
Example of 1-B

GIVEN the set of ordered pairs:

$$\{(-3, 2), (4, -1), (5, 2), (-1, -3), (-4, 5)\}$$

IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

Solution:



Example of 1-C

GIVEN the equation

$$y = |x - 3|$$

with $x = -2$,

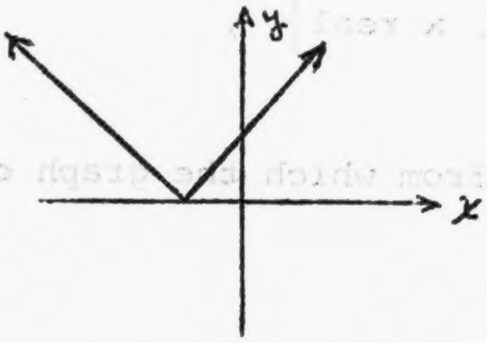
COMPUTE the corresponding value of y.

Solution: $y = |-2 -3| = |-5| = 5$

Example of 1-D

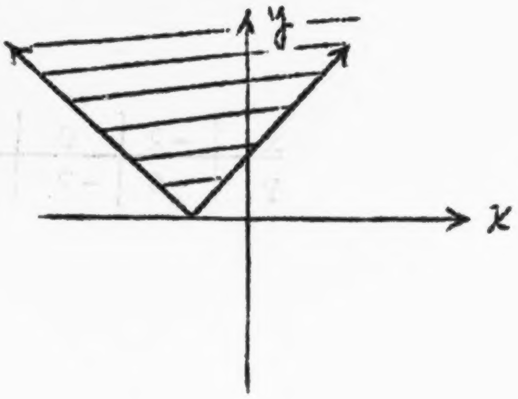
GIVEN the following curve for the relation

$$\{(x, y) \mid y \geq |x + 2| , x \text{ real}\}$$



IDENTIFY by shading the region which satisfies the inequality for the relation.

Solution:



Example of 2-A

GIVEN the relation

$$\left\{ (x, y) \;\middle|\; y \geq x - 2,\ x \text{ real} \right\} ,$$

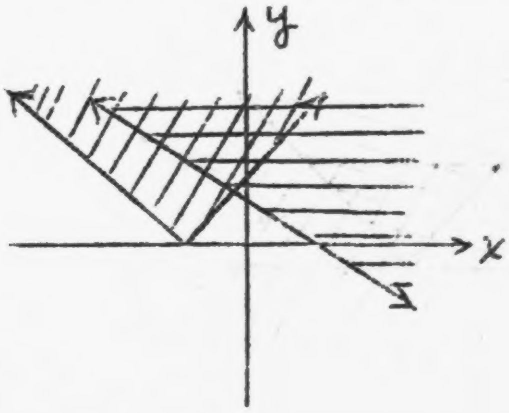
CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

Solution:

x	-2	0	2
y	-4	-2	0

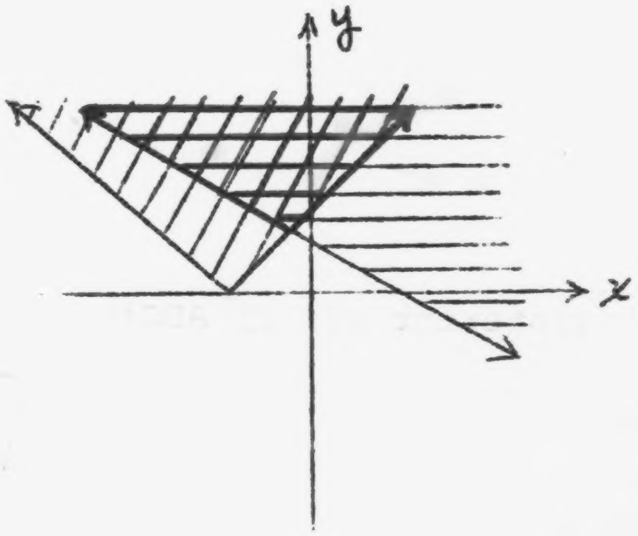
Example of 2-B

GIVEN the following graphs of two different relations:



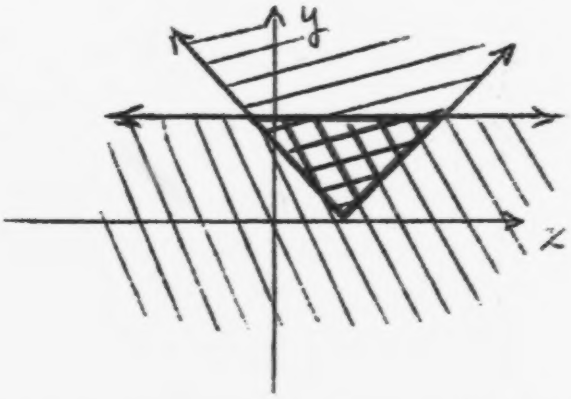
IDENTIFY their $\cap$ by shading in red.

Solution:

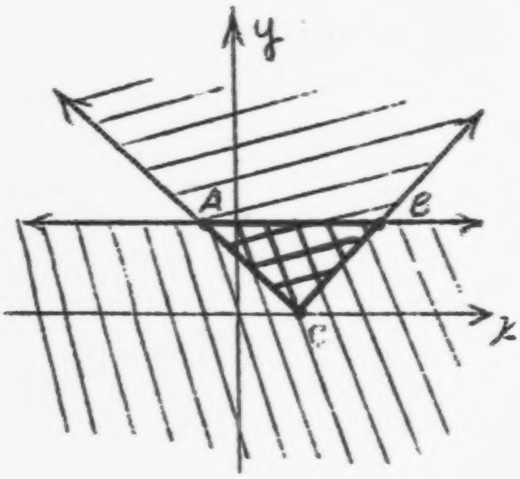


Example of 3-A

GIVEN the following graphs of two different relations with the intersection of the two graphs shown in red:



NAME the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.



Solution:

△ ABC ∪ (Interior of △ ABC)

Example of 3-B

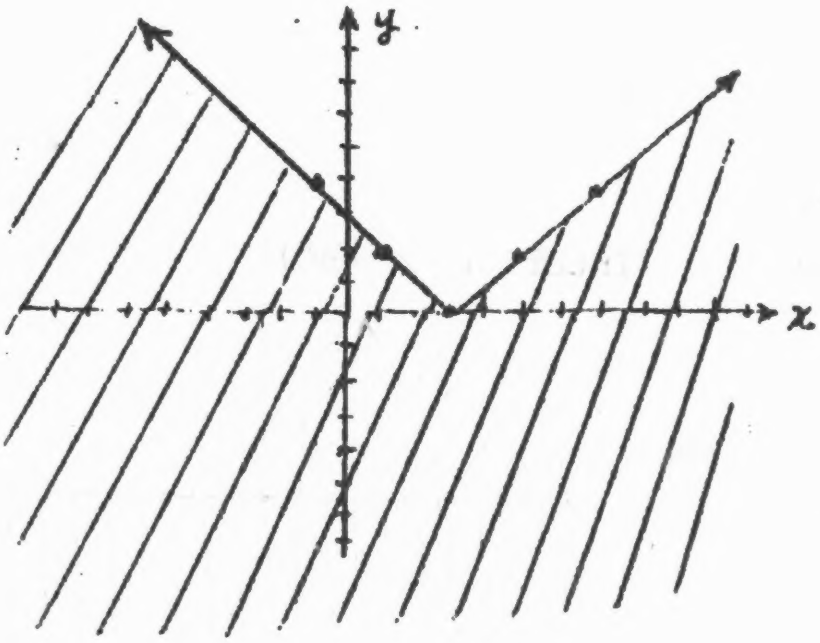
GIVEN the relation

$$\left\{ (x, y) \,\middle|\, y \le |x - 3| \text{ , x real} \right\},$$

CONSTRUCT a graph of its solution set.

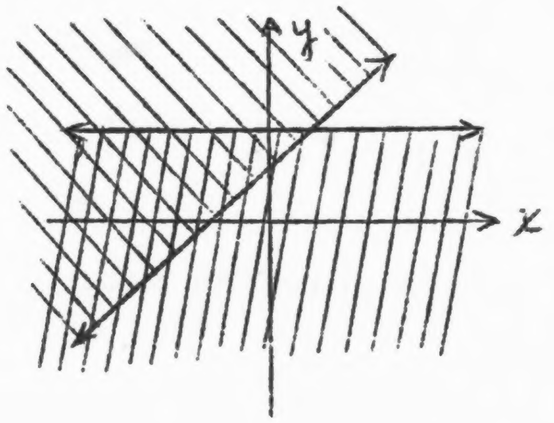
Solution:

x	-1	1	3	5	7
y	4	2	0	2	4



Example of 3-C

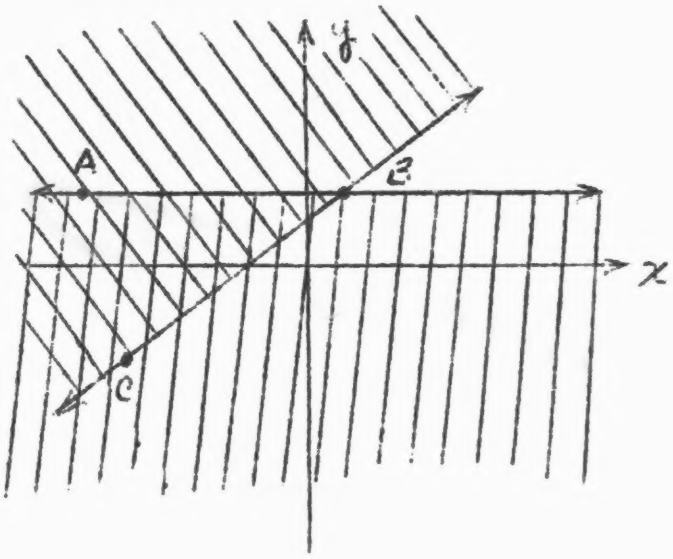
GIVEN the following graph of a system of two relations:



NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

$\angle$ ABC $\cup$ (Interior $\angle$ ABC)



Example of 4-A

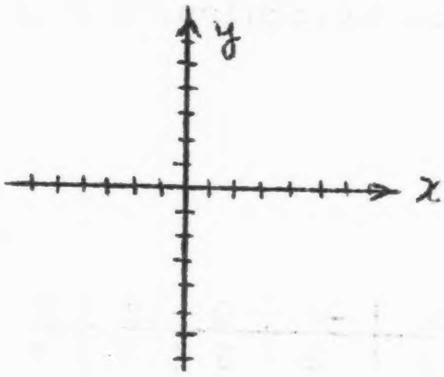
GIVEN the following two relations

$$\{(x, y) \mid y \geq x - 1, \text{ x real}\}$$

and

$$\{(x, y) \mid y < 2 - x, \text{ x real}\}$$

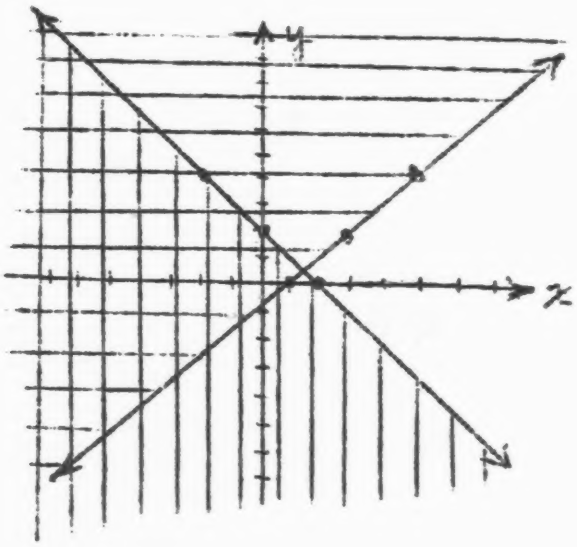
CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



Solution:

x	1	3	5
y	0	2	4

x	-2	0	2
y	4	2	0



Example of 4-B

GIVEN the following system of two relations

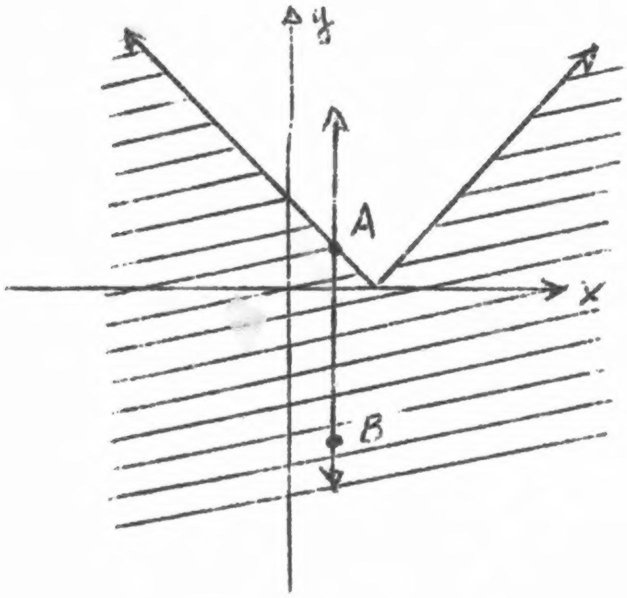
$$\begin{cases} \{(x, y) \mid y \leq x - 3, \ x \text{ real}\} \\ \{(x, y) \quad x = 2, \ x \text{ real}\} \end{cases}$$

CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

x	-3	0	3	6
y	6	3	0	3

$\overrightarrow{AB}$



Example of 5-A

The students in Treatment 3 received the same text material at this point as students in the control group (Treatment 1). This insert represents the point in the treatment at which the following text materials were administered:

Activity I
Activity II
Activity III
Activity IV
Activity V
Activity VI
Activity VIIa
Activity VIIb
Activity VIIIa
Activity VIIIb

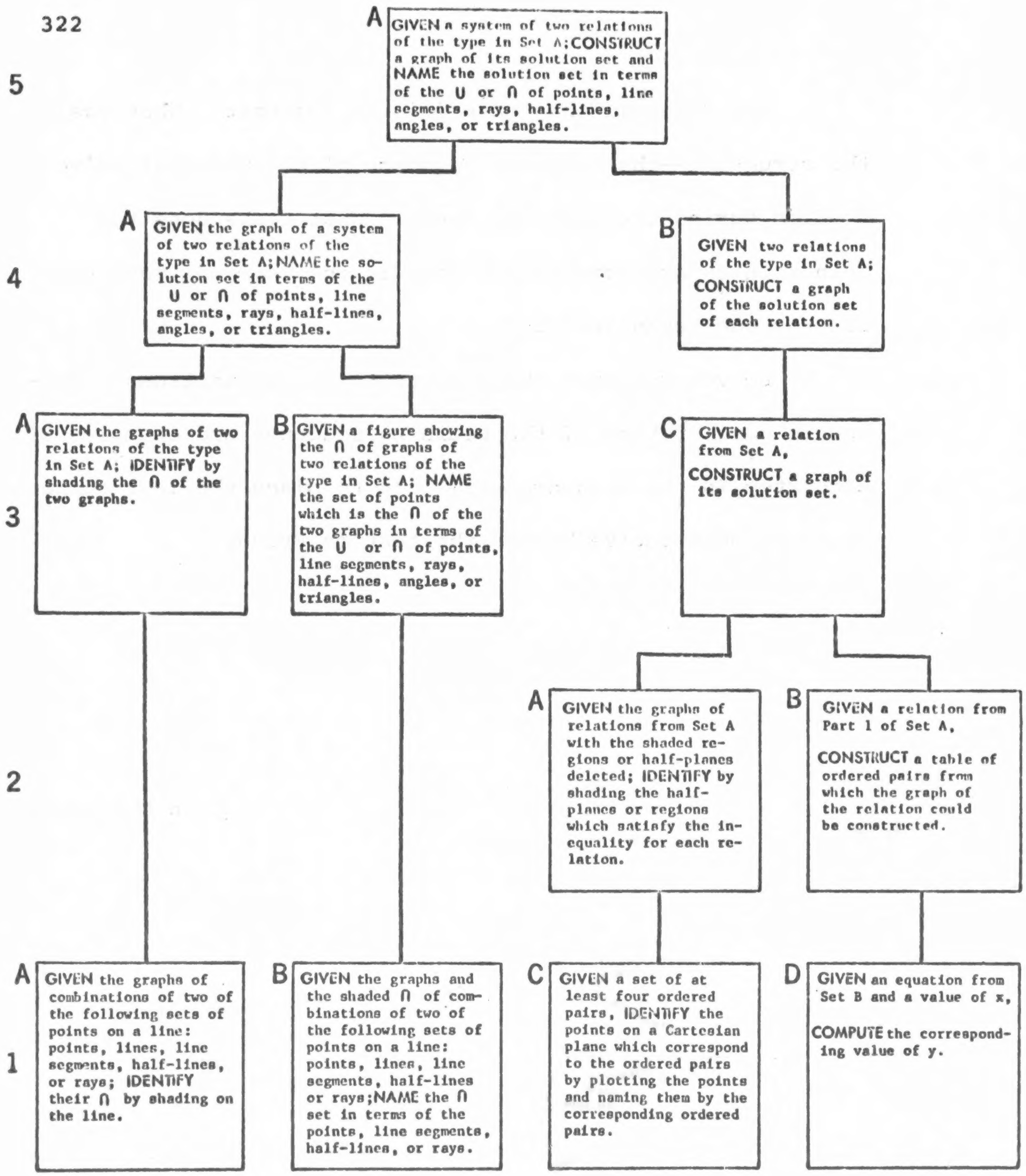
The following review material was given to the students in Treatment 3 at the completion of the "check-up" quiz for activity VIIIb.

How does all this material fit together? What was the structure--the Learning Sequence of the material we've studied during the last few weeks? What tasks does the author think are important? What is the terminal task that all the sub-tasks lead to?

Do you remember the answers to these questions? Perhaps a quick review of the Learning Sequence and examples of each step in the sequence will help to place the last three weeks in perspective before the test on Friday.

Turn the page.

LEARNING SEQUENCE



5 A **GIVEN** a system of two relations of the type in Set A; **CONSTRUCT** a graph of its solution set and **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4 A **GIVEN** the graph of a system of two relations of the type in Set A; **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4 B **GIVEN** two relations of the type in Set A; **CONSTRUCT** a graph of the solution set of each relation.

3 A **GIVEN** the graphs of two relations of the type in Set A; **IDENTIFY** by shading the ∩ of the two graphs.

3 B **GIVEN** a figure showing the ∩ of graphs of two relations of the type in Set A; **NAME** the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

3 C **GIVEN** a relation from Set A, **CONSTRUCT** a graph of its solution set.

2 A **GIVEN** the graphs of relations from Set A with the shaded regions or half-planes deleted; **IDENTIFY** by shading the half-planes or regions which satisfy the inequality for each relation.

2 B **GIVEN** a relation from Part 1 of Set A, **CONSTRUCT** a table of ordered pairs from which the graph of the relation could be constructed.

1 A **GIVEN** the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; **IDENTIFY** their ∩ by shading on the line.

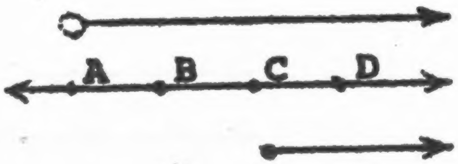
1 B **GIVEN** the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; **NAME** the ∩ set in terms of the points, line segments, half-lines, or rays.

1 C **GIVEN** a set of at least four ordered pairs, **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

1 D **GIVEN** an equation from Set B and a value of x, **COMPUTE** the corresponding value of y.

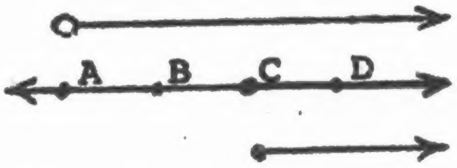
Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert ,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$ Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$ Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$ Note: a, b, and c are integers.

GIVEN the line $\overleftrightarrow{AD}$. The graph of one set of points ($\overrightarrow{CD}$) on $\overleftrightarrow{AD}$ is shown below $\overleftrightarrow{AD}$ and the graph of a second set of points (half-line AD) on $\overleftrightarrow{AD}$ is shown above $\overleftrightarrow{AD}$.



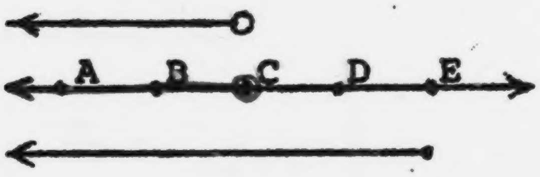
IDENTIFY the intersection of $\overrightarrow{CD}$ and the half-line AD by shading in red on line $\overleftrightarrow{AD}$.

Solution:



Example of 1-A

GIVEN the line $\overleftrightarrow{AE}$. The graph of the set of points half-line CA on $\overleftrightarrow{AE}$ is shown above $\overleftrightarrow{AE}$ and the set of points $\overrightarrow{EA}$ on $\overleftrightarrow{AE}$ is shown below $\overleftrightarrow{AD}$. The $\cap$ of half-line CA and $\overrightarrow{EA}$ is shown in red on $\overleftrightarrow{AE}$.



NAME the $\cap$ set in terms of points, line segments, half-lines, or rays.

Solution: half-line CA

Example of 1-B

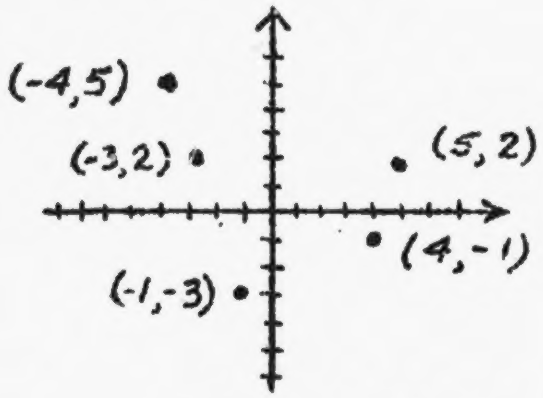
6

GIVEN the set of ordered pairs:

$$\left\{(-3, 2), (4, -1), (5, 2), (-1, -3), (-4, 5)\right\}$$

IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

Solution:



Example of 1-C

GIVEN the equation

$$y = |x - 3|$$

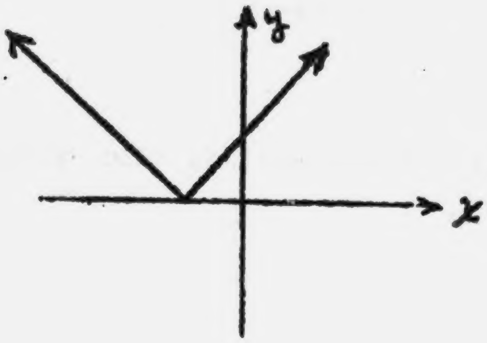
with $x = -2$,

COMPUTE the corresponding value of y.

Solution: $y = |-2 -3| = |-5| = 5$

Example of 1-D

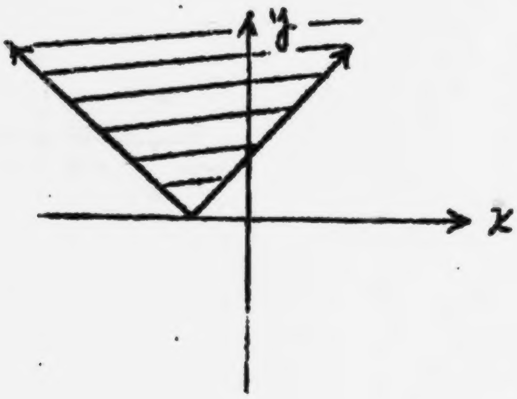
GIVEN the following curve for the relation



$\{(x, y) \mid y \geq |x + 2|, x \text{ real}\}$

IDENTIFY by shading the region which satisfies the inequality for the relation.

Solution:



Example of 2-A

GIVEN the relation

$$\left\{ (x,\ y)\ \middle|\ y \geq x - 2,\ x \text{ real} \right\} ,$$

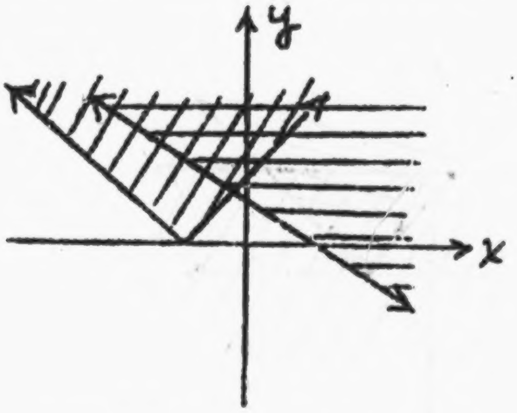
CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

Solution:

x	-2	0	2
y	-4	-2	0

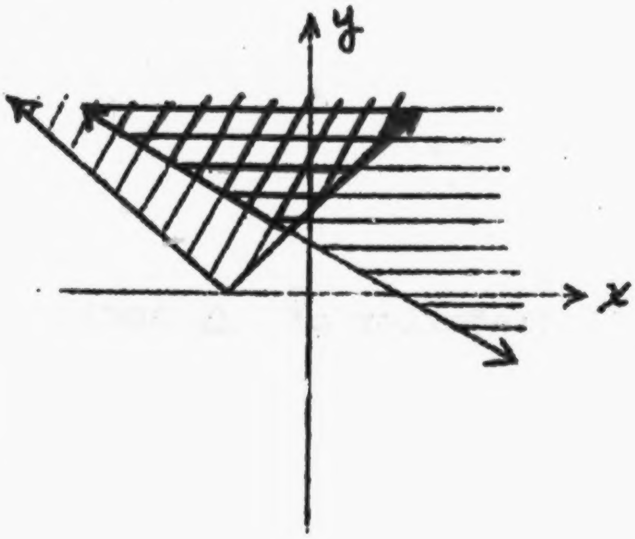
Example of 2-B

GIVEN the following graphs of two different relations:



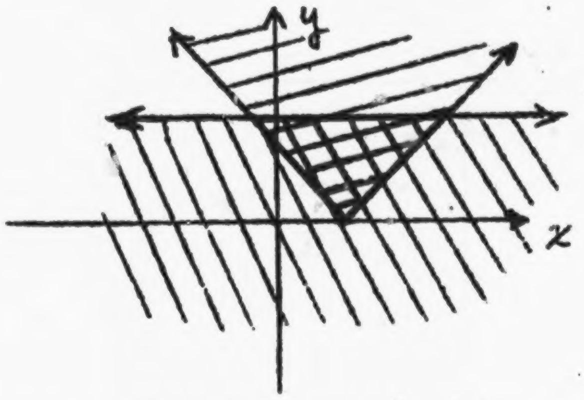
IDENTIFY their $\cap$ by shading in red.

Solution:



Example of 3-A

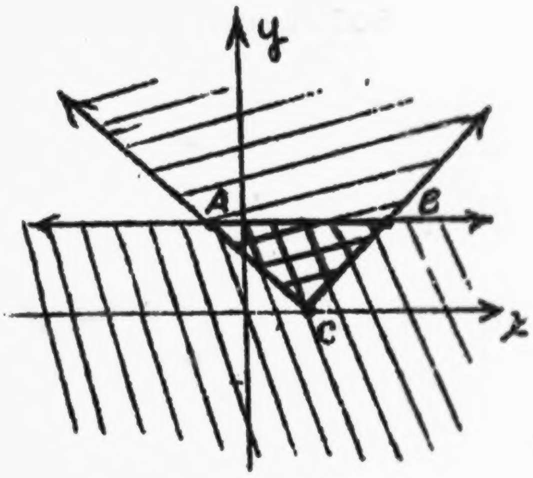
GIVEN the following graphs of two different relations with the intersection of the two graphs shown in red:



NAME the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

$\triangle$ ABC $\cup$ (Interior of $\triangle$ ABC)



Example of 3-B

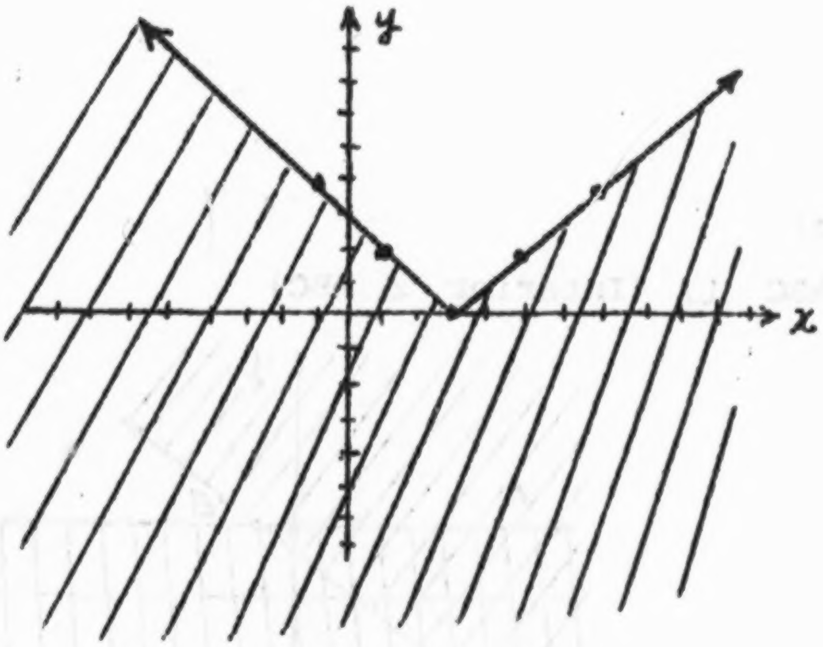
GIVEN the relation

$$\left\{(x, y) \,\middle|\, y \leq |x - 3| \;, \; x \text{ real}\right\},$$

CONSTRUCT a graph of its solution set.

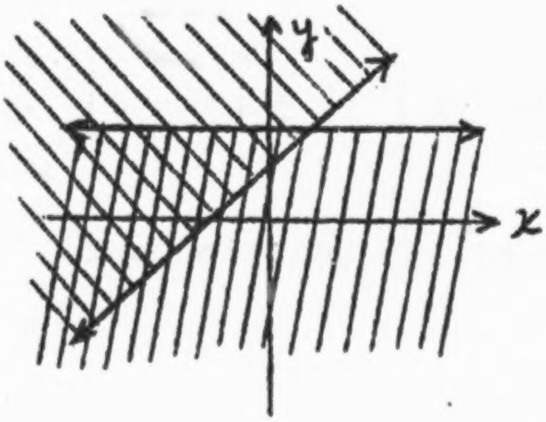
Solution:

x	-1	1	3	5	7
y	4	2	0	2	4



Example of 3-C

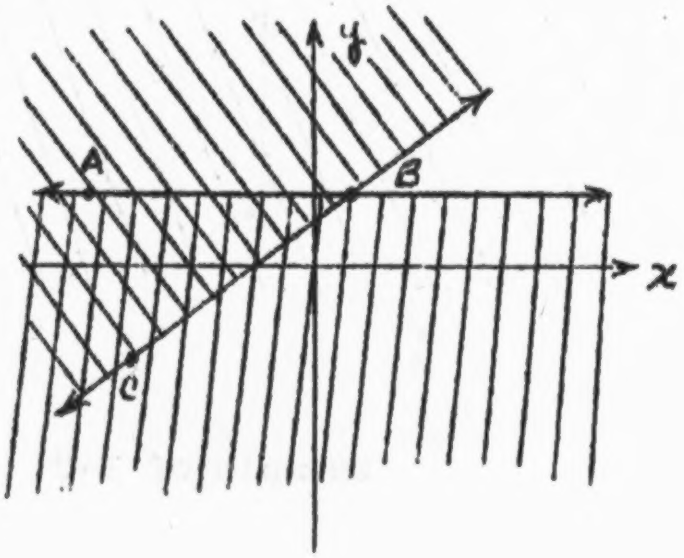
GIVEN the following graph of a system of two relations:



NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

$\angle$ ABC $\cup$ (Interior $\angle$ ABC)



Example of 4-A

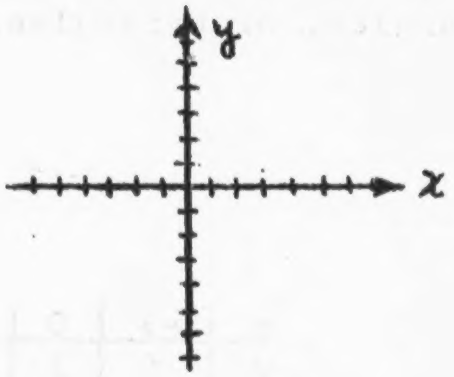
GIVEN the following two relations

$$\left\{ (x, y) \mid y \geq x - 1, x \text{ real} \right\}$$

and

$$\left\{ (x, y) \mid y < 2 - x, x \text{ real} \right\}$$

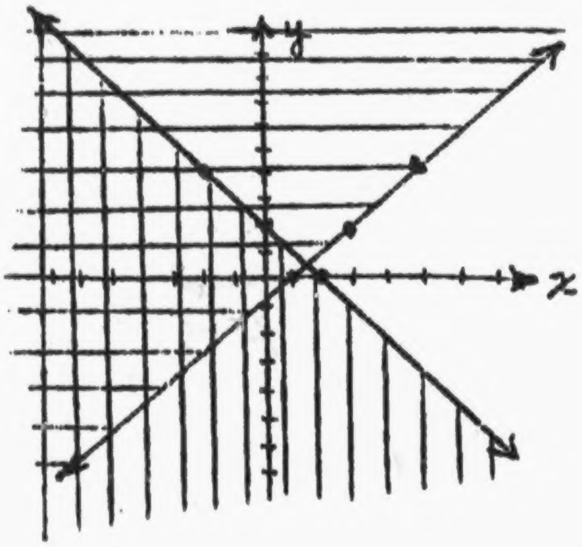
CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



Solution:

x	1	3	5
y	0	2	4

x	-2	0	2
y	4	2	0



Example of 4-B

GIVEN the following system of two relations

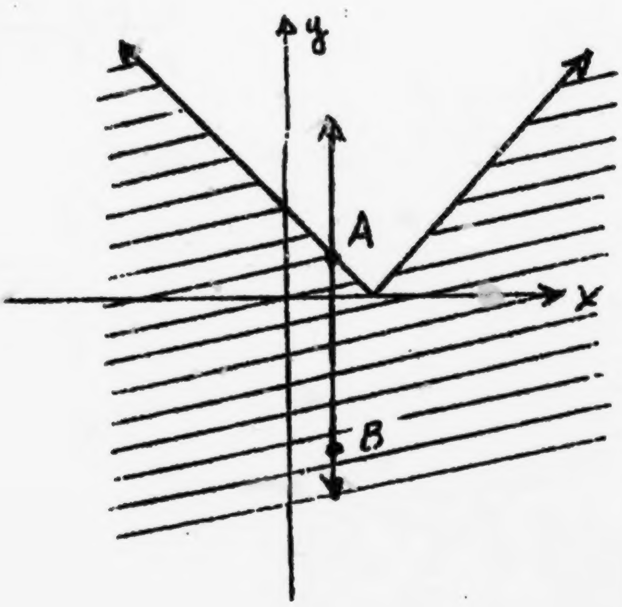
$$\begin{cases} \{(x, y) \mid y \leq x - 3 \quad , x \text{ real}\} \\ \{(x, y) \quad x = 2, x \text{ real}\} \end{cases}$$

CONSTRUCT a graph of its solution set and **NAME** the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

x	-3	0	3	6
y	6	3	0	3

$\overrightarrow{AB}$



Example of 5-A

APPENDIX D

INTRODUCTION, LEARNING HIERARCHY INFORMATION AT THE BEGINNING OF THE UNIT OF INSTRUCTION, BEHAVIORAL OBJECTIVES OF EACH ACTIVITY, AND EACH ACTIVITY'S PLACE IN THE LEARNING HIERARCHY, AND TEXT MATERIAL INSERTS (TREATMENT T_4)

INTRODUCTION

The materials for your next unit of instruction were written by the University of Maryland Mathematics Project. Perhaps you will find that the format of the materials will differ from the textbooks that you have previously used. Ordinary textbooks usually require homework. Homework is not required for this material. In this material, all your work will be done individually in class. You will be asked to answer questions by writing on the blanks provided throughout the materials or by performing the other tasks requested.

These materials are designed around _what you do_. If you work on each task at the point it is called for, you will find that the materials have a completeness about them that is not in an ordinary textbook. You will also find that your response at each step will help you to acquire the skills upon which you will be tested. FOR YOUR BENEFIT _do not read further_ until you have completed each task the way you think it should be done. (No one is going to count the number of questions answered correctly--_the tasks are for your benefit_.) If you answer incorrectly or perform a task incorrectly, re-read the questions and correct your original response.

At the beginning of each class, your instructor will inform you of the maximum number of lessons which you will be permitted to complete that day. Upon completing a lesson, raise your hand and you will be given another lesson. When you have completed all the lessons permitted for that day, raise your hand and you will be given your "check-up questions." Upon completing your "check-up questions," raise your hand and the instructor will pick up your questions. If you complete the check-up questions before the end of the class period, you may work on some reading assignment. Although everyone will be working at his own pace, it is suggested that you take your time.

You are expected to make arrangements with your instructor for any lesson you might miss because of absence.

Here are some examples to show you how the format will look.

Sets of elements are denoted by the two braces $\{$ and $\}$. Given the two sets of integers

$$\{1, 2, 3, 4, 5\} \quad \text{and} \quad \{4, 6, 7\},$$

circle in red that integer which is common to both sets. Do not read further until you have performed the task!

Did you circle the 5? You didn't? You are correct, the 5 is not in both sets. However, the 4 is in both sets and you are correct if you circled the 4 in both sets.

Given the following two sets

$$\{1, 2, 3\} \quad \text{and} \quad \{4, 5\} \text{ ,}$$

list all the integers which are in either of the two sets: __________. Have you listed the integers? Do it now before they disappear--or before you read further.

Did you list only the integers 1, 2, and 3? If you did, re-read the question. You were asked to list the integers in either of the two sets. The integers which are in either of the two sets are 1, 2, 3, 4, and 5.

Turn the page. You are ready to begin.

LEARNING SEQUENCE

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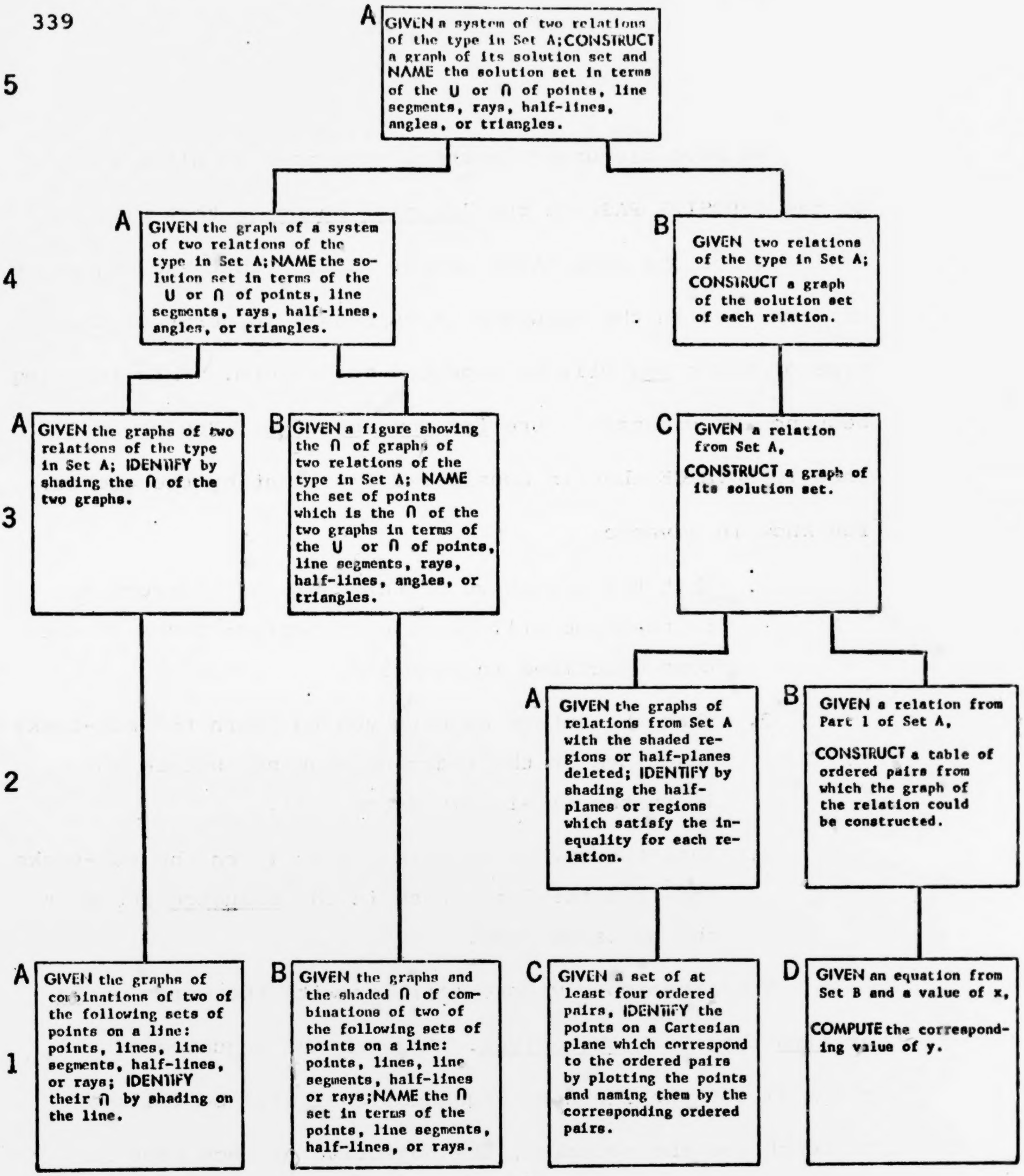


5 A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4 A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4 B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3 A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

3 B GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

3 C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2 A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

2 B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1 A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

1 B GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

1 C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

1 D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x+a \rvert, x \text{ real}\}$ $\{(x,y) \mid y \geq x+b, x \text{ real}\}$ $\{(x,y) \mid y \geq c-x, x \text{ reals}\}$ Part 2. $\{(x,y) \mid y \geq d, x \text{ real}\}$ $\{(x,y) \mid x \geq e, x \text{ real}\}$ Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	$y = \lvert x+a \rvert$ $y = x+b$ $y = c-x$ Note: a, b, and c are integers.

We have discussed learning sequences in class today. On the OPPOSITE PAGE is the learning sequence that will be followed for the next three weeks. Notice that the objective of each step in the sequence is expressed in terms of the type of tasks you will be expected to perform. This learning sequence is presented here for your benefit. You now can know IN ADVANCE what is considered important by the author. You know in advance:

1. That the objective of this unit of instruction is that you will be able to perform tasks of the type described in step 5-A.
2. That the author expects you to learn the sub-tasks described in the Learning Sequence before you learn the final task (step 5-A).
3. That the author expects you to learn the sub-tasks and then the final task in the sequence shown on the opposite page.

You are probably more familiar with the words goals or aims than with objectives. As with the sequence of steps (i.e., the learning sequence) that will be followed to teach you the material, the objective of each step in the sequence is for your benefit. The objective of a step is a description of what a successful learner will be able to do when he has completed the lesson for that step.

Awareness of these objectives at the beginning of this unit of instruction will enable you to direct your efforts during the next three weeks toward learning how to perform the specific tasks described by the objectives. You now know what you must do to succeed:

> You are to learn to perform the sub-tasks in the sequence shown so that you will be able to perform the final task.

The author is very much aware that you may not be familiar with the terms used in the statements of objectives in the learning sequence. Hence, FOR YOUR BENEFIT, examples of the type of tasks you're expected to learn to perform at each step are provided in the green folder which you have been given. Take time now to look at the examples and see how they match the statements of the objectives in the Learning Sequence. If you take time now, you will save yourself a lot of work later.

Note that Set A and Set B referred to in several places in the sequence are shown at the bottom of the opposite page.

At the start of each lesson you will receive:

1. A statement of the objective for the lesson;

and 2. An example of the objective, i.e., of the type of task you will be expected to perform at the end of the lesson.

Take careful note of the lesson's objective; your awareness of the objective will be checked at the end of the lesson. In addition, you will be told at the beginning of each lesson:

3. The place of the lesson in the Learning Sequence. With this third piece of information, you will know why you will be able to learn the task of the new lesson and why the author wants you to learn to perform the task.

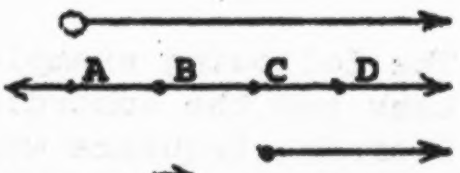
There is no statement of objective for Lesson I. Lesson I is simply provided to help you to remember some concepts about sets. There is a definite reason that the statement of the objective was not provided. By not stating an objective for Lesson I, the author is telling you something very important. You will not be expected to remember anything specifically from Lesson I.

Wait! Stop! Have you read carefully the Learning Sequence and the examples? Don't rush down a road until you know where the road leads.

Turn the page.

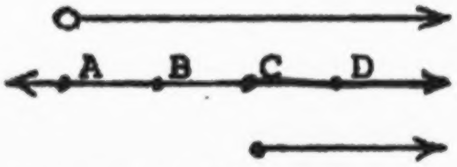
The following examples of the terminal task and the subordinate tasks in the Learning Sequence were given to the students at this point.

GIVEN the line $\overleftrightarrow{AD}$. The graph of one set of points ($\overrightarrow{CD}$) on $\overleftrightarrow{AD}$ is shown below $\overleftrightarrow{AD}$ and the graph of a second set of points (half-line AD) on $\overleftrightarrow{AD}$ is shown above $\overleftrightarrow{AD}$.



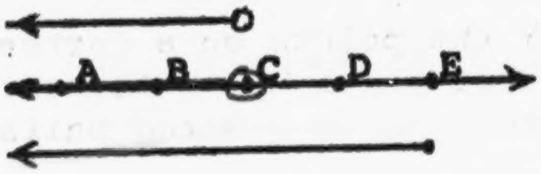
IDENTIFY the intersection of $\overrightarrow{CD}$ and the half-line AD by shading in red on line $\overleftrightarrow{AD}$.

Solution:



Example of 1-A

GIVEN the line $\overleftrightarrow{AE}$. The graph of the set of points half-line CA on $\overleftrightarrow{AE}$ is shown above $\overleftrightarrow{AE}$ and the set of points $\overrightarrow{EA}$ on $\overleftrightarrow{AE}$ is shown below $\overleftrightarrow{AD}$. The $\cap$ of half-line CA and $\overrightarrow{EA}$ is shown in red on $\overleftrightarrow{AE}$.



NAME the $\cap$ set in terms of points, line segments, half-lines, or rays.

Solution: half-line CA

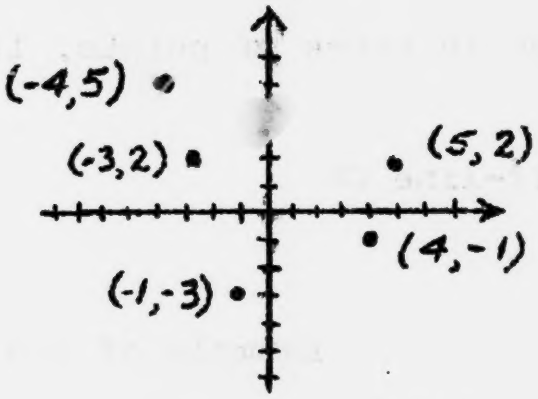
Example of 1-B

GIVEN the set of ordered pairs:

$$\{(-3,\ 2),\ (4,\ -1),\ (5,\ 2),\ (-1,\ -3),\ (-4,\ 5)\}$$

IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

Solution:



Example of 1-C

GIVEN the equation

$$y = |x - 3|$$

with $x = -2$,

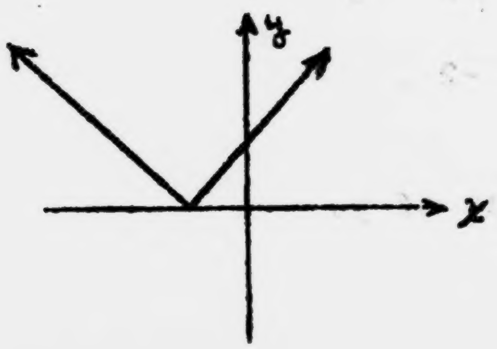
COMPUTE the corresponding value of y.

Solution: $y = |-2 - 3| = |-5| = 5$

Example of 1-D

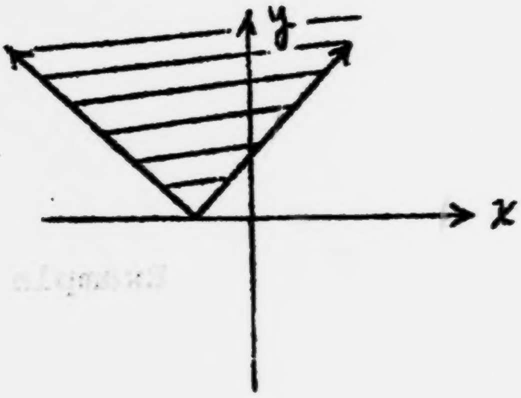
GIVEN the following curve for the relation

$$\left\{ (x, y) \;\middle|\; y \geq |x + 2| \;,\; x \text{ real} \right\}$$



IDENTIFY by shading the region which satisfies the inequality for the relation.

Solution:



Example of 2-A

GIVEN the relation

$$\left\{(x,\ y)\ \middle|\ y \geq x - 2,\ x \text{ real}\right\},$$

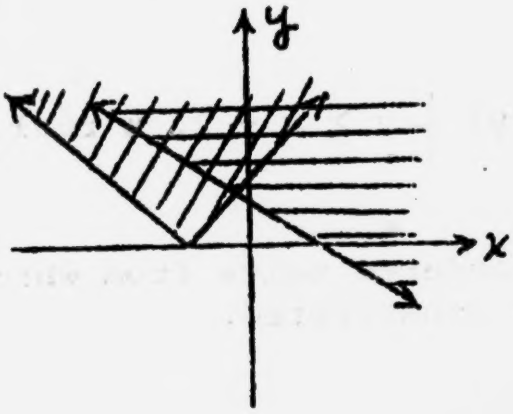
CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

Solution:

x	-2	0	2
y	-4	-2	0

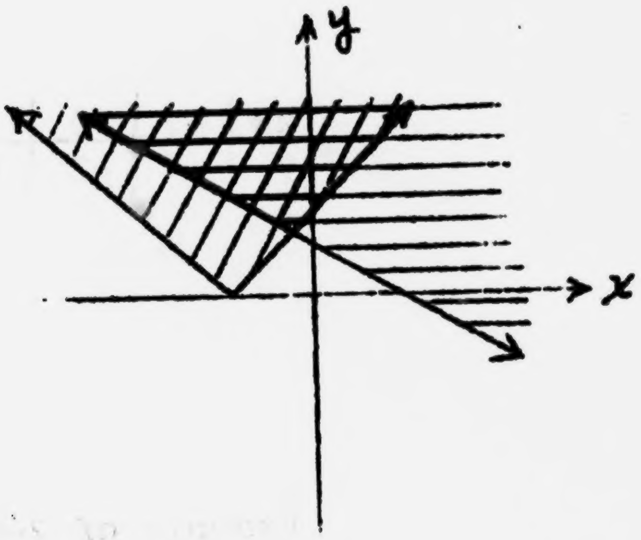
Example of 2-B

GIVEN the following graphs of two different relations:



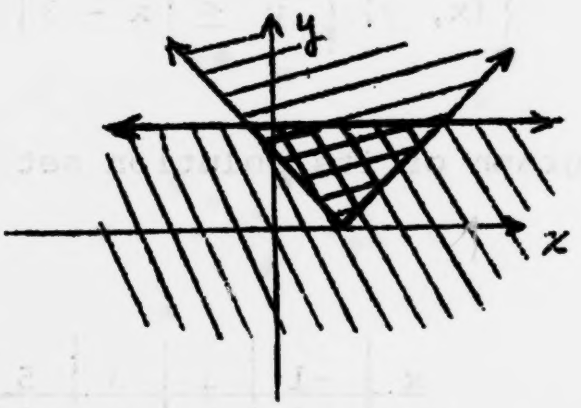
IDENTIFY their $\cap$ by shading in red.

Solution:



Example of 3-A

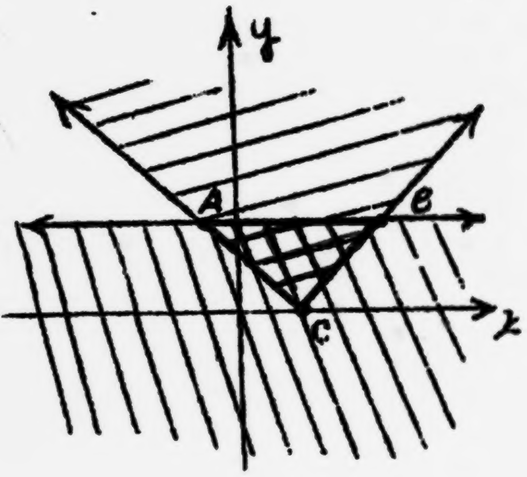
GIVEN the following graphs of two different relations with the intersection of the two graphs shown in red:



NAME the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

△ ABC ∪ (Interior of △ ABC)



Example of 3-B

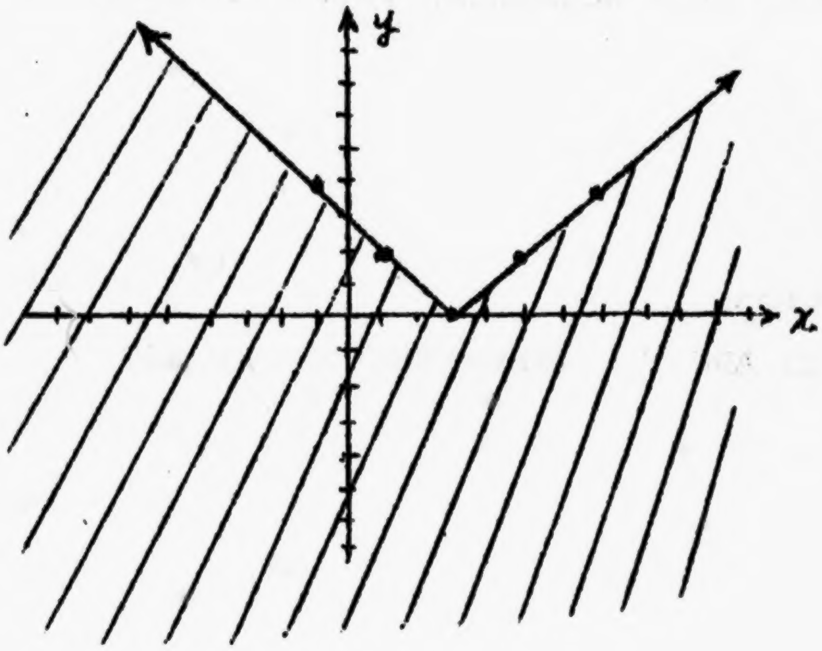
GIVEN the relation

$$\left\{(x, y) \;\middle|\; y \le |x - 3| \;, \; x \text{ real}\right\},$$

CONSTRUCT a graph of its solution set.

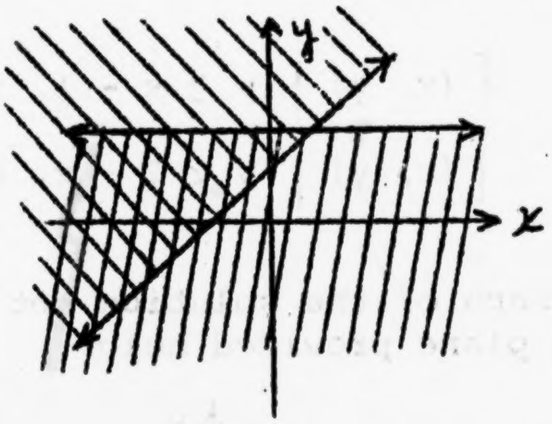
Solution:

x	-1	1	3	5	7
y	4	2	0	2	4



Example of 3-C

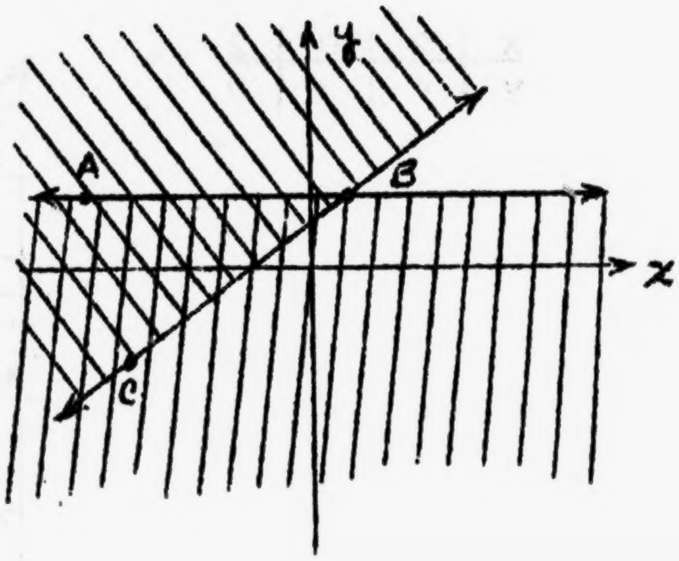
GIVEN the following graph of a system of two relations:



NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays; half-lines, angles, or triangles.

Solution:

$\angle$ ABC $\cup$ (Interior $\angle$ ABC)



Example of 4-A

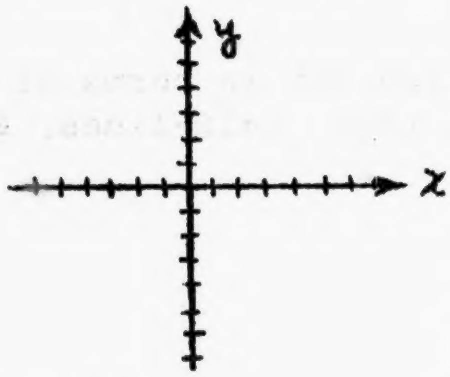
GIVEN the following two relations

$$\left\{ (x, y) \;\middle|\; y \geq x - 1,\ x \text{ real} \right\}$$

and

$$\left\{ (x, y) \;\middle|\; y < 2 - x,\ x \text{ real} \right\}$$

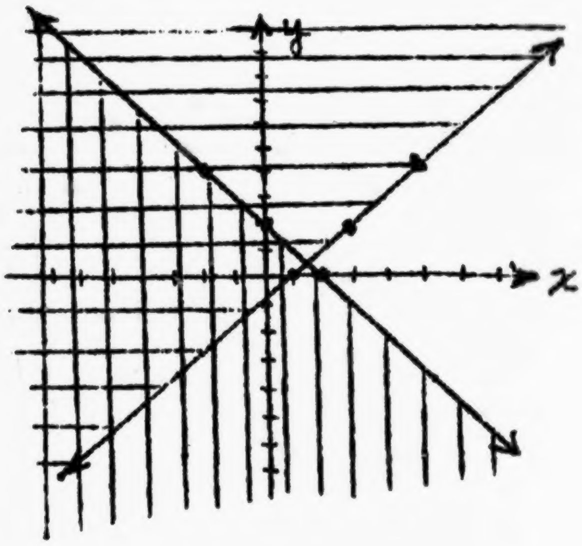
CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



Solution:

x	1	3	5
y	0	2	4

x	-2	0	2
y	4	2	0



Example of 4-B

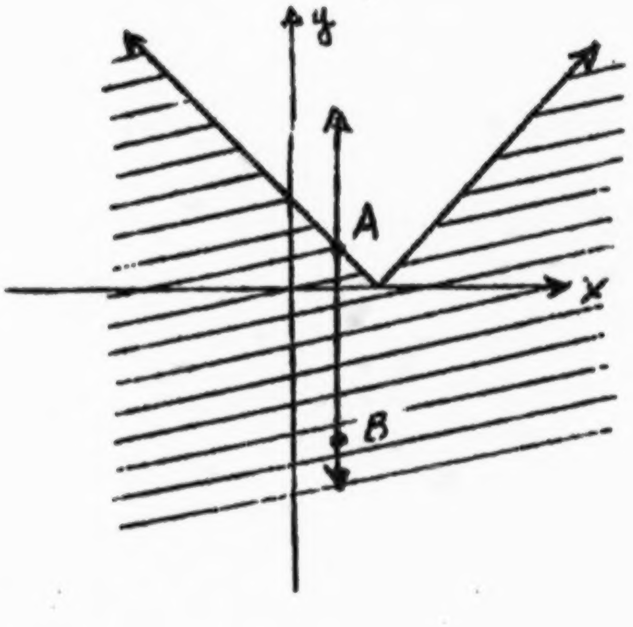
GIVEN the following system of two relations

$$\begin{cases}\{(x,\ y) \mid y \leq x - 3\ ,\ x \text{ real}\} \\ \{(x,\ y)\quad x = 2,\ x \text{ real}\}\end{cases}$$

CONSTRUCT a graph of its solution set and **NAME** the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

x	-3	0	3	6
y	6	3	0	3



$\overrightarrow{AB}$

Example of 5-A

Activity I text material insert.

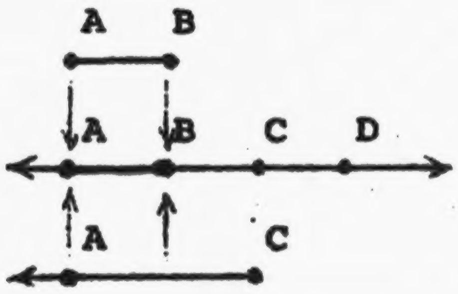
What Do I Have to Learn in This Lesson?

At the end of Lesson II you should be able to perform the following task:

> **GIVEN the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays;**
>
> **NAME the $\cap$ set in terms of the points, line segments, half-lines, or rays.**

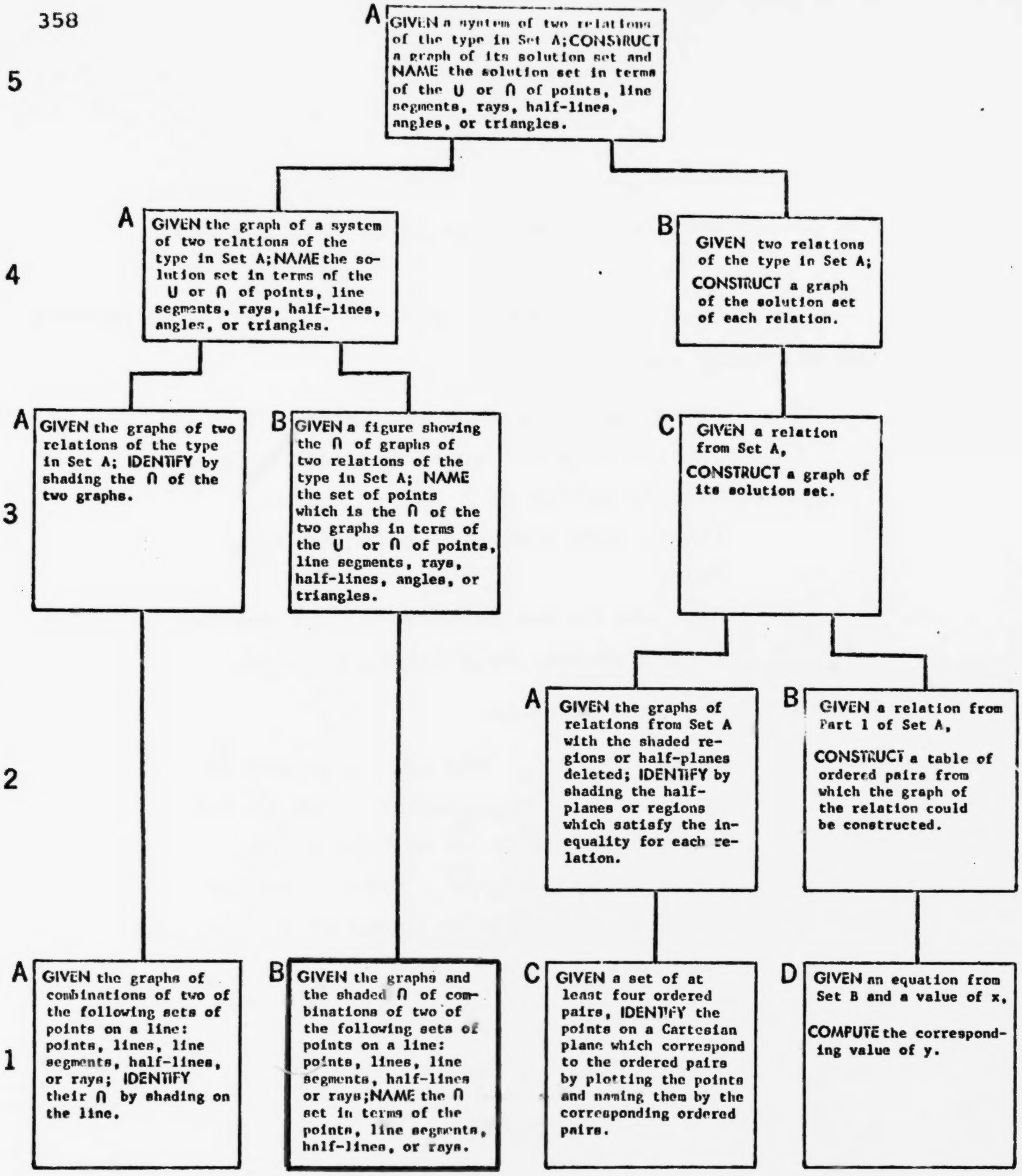
Example of the type of task:

> **GIVEN line $\overleftrightarrow{AC}$. The set of points $\overline{AB}$ on line $\overleftrightarrow{AC}$ is represented above $\overleftrightarrow{AC}$ and the set of points $\overrightarrow{CA}$ on line $\overleftrightarrow{AC}$ is represented below $\overleftrightarrow{AC}$. The $\cap$ of the two sets of points is shown in red on line $\overleftrightarrow{AC}$.**



> **NAME the $\cap$ set (shown in red) of the two sets of points $\overline{AB}$ and $\overrightarrow{CA}$.**
>
> **Answer: $\overline{AB}$**

LEARNING SEQUENCE



5

A **GIVEN** a system of two relations of the type in Set A; **CONSTRUCT** a graph of its solution set and **NAME** the solution set in terms of the U or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A **GIVEN** the graph of a system of two relations of the type in Set A; **NAME** the solution set in terms of the U or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B **GIVEN** two relations of the type in Set A; **CONSTRUCT** a graph of the solution set of each relation.

3

A **GIVEN** the graphs of two relations of the type in Set A; **IDENTIFY** by shading the ∩ of the two graphs.

B **GIVEN** a figure showing the ∩ of graphs of two relations of the type in Set A; **NAME** the set of points which is the ∩ of the two graphs in terms of the U or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C **GIVEN** a relation from Set A, **CONSTRUCT** a graph of its solution set.

2

A **GIVEN** the graphs of relations from Set A with the shaded regions or half-planes deleted; **IDENTIFY** by shading the half-planes or regions which satisfy the inequality for each relation.

B **GIVEN** a relation from Part 1 of Set A, **CONSTRUCT** a table of ordered pairs from which the graph of the relation could be constructed.

1

A **GIVEN** the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; **IDENTIFY** their ∩ by shading on the line.

B **GIVEN** the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; **NAME** the ∩ set in terms of the points, line segments, half-lines, or rays.

C **GIVEN** a set of at least four ordered pairs, **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D **GIVEN** an equation from Set B and a value of x, **COMPUTE** the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert, x \text{ real}\}$ $\{(x,y) \mid y \geq x + b, x \text{ real}\}$ $\{(x,y) \mid y \geq c - x, x \text{ reals}\}$	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$
Part 2. $\{(x,y) \mid y \geq d, x \text{ real}\}$ $\{(x,y) \mid x \geq e, x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=, >, <$, or $\leq$; a, b, c, d, and e are integers.	

You now are ahead of the game! You know the type of problem the author considers important. Obviously, this is the type of problem you might be tested upon. Pay particular attention when he discusses how to name the $\cap$ set of lines, rays, etc.

The place of this lesson is shown in red on the Learning Sequence. Lesson II is step 1-B in the Learning Sequence. The author wants you to learn to perform the task of Lesson II so that you will have the necessary skill to learn to perform the task of step 3-B.

Turn the page.

Activity II text material insert.

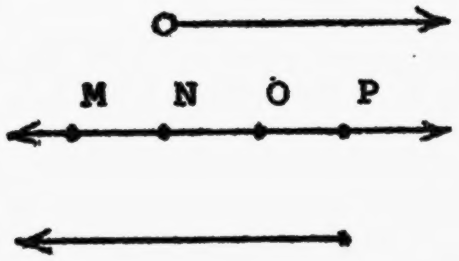
Watch Out For the Objective of Lesson III!

The goal or the objective of Lesson III is to teach you to perform the following task:

> GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays;
>
> IDENTIFY their $\cap$ by shading on the line.

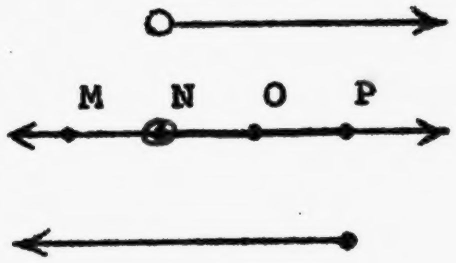
Example of the type of task:

> GIVEN line $\overleftrightarrow{MP}$. One set of points on line $\overleftrightarrow{MP}$ is represented above $\overleftrightarrow{MP}$ while a second set of points on $\overleftrightarrow{MP}$ is represented below $\overleftrightarrow{MP}$.

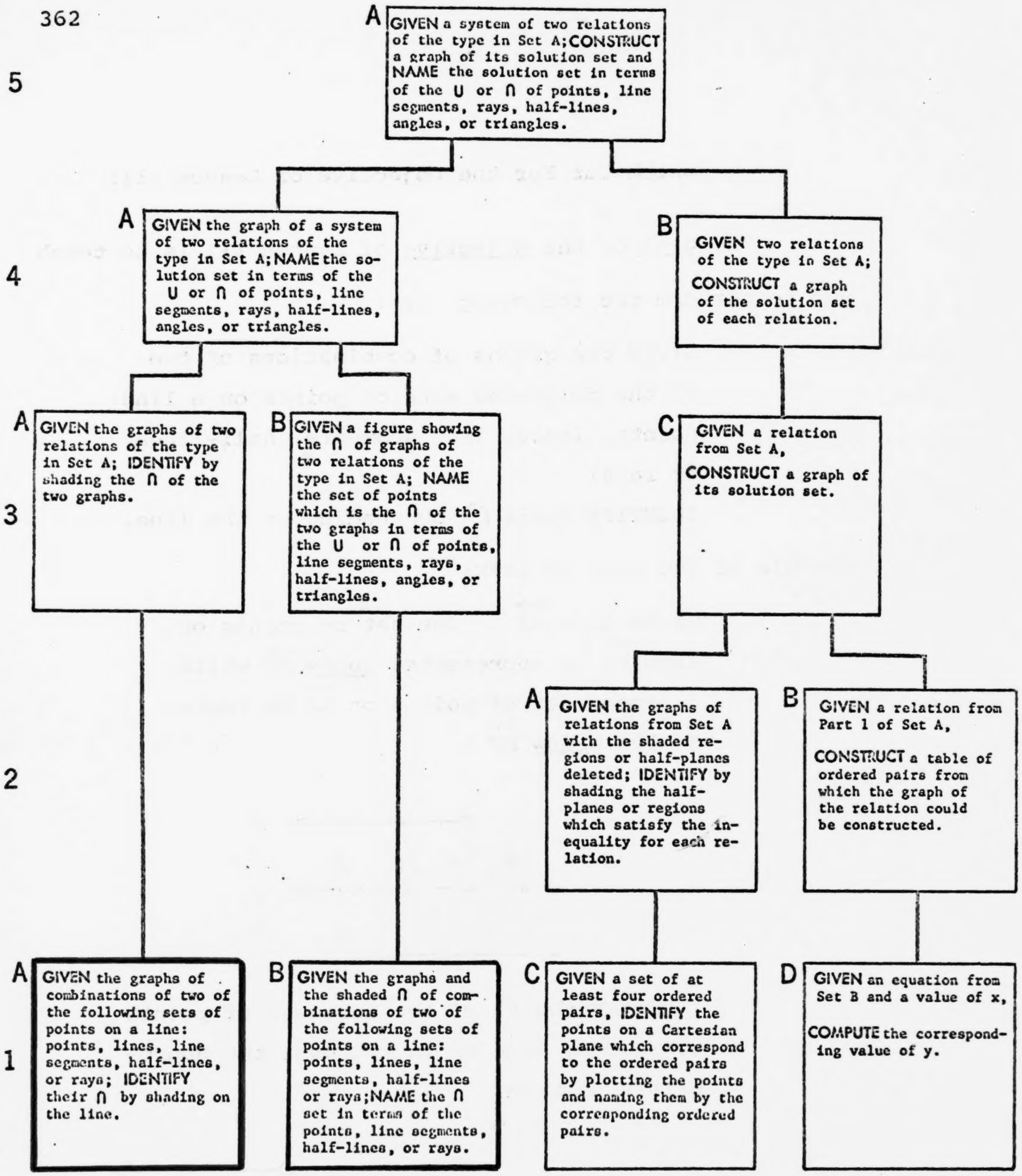


> IDENTIFY the $\cap$ of the two sets of points by shading line $\overleftrightarrow{MP}$ in red over the appropriate portion.

Solution:



LEARNING SEQUENCE



5

A **GIVEN** a system of two relations of the type in Set A; **CONSTRUCT** a graph of its solution set and **NAME** the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4

A **GIVEN** the graph of a system of two relations of the type in Set A; **NAME** the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

B **GIVEN** two relations of the type in Set A; **CONSTRUCT** a graph of the solution set of each relation.

3

A **GIVEN** the graphs of two relations of the type in Set A; **IDENTIFY** by shading the $\cap$ of the two graphs.

B **GIVEN** a figure showing the $\cap$ of graphs of two relations of the type in Set A; **NAME** the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

C **GIVEN** a relation from Set A, **CONSTRUCT** a graph of its solution set.

2

A **GIVEN** the graphs of relations from Set A with the shaded regions or half-planes deleted; **IDENTIFY** by shading the half-planes or regions which satisfy the inequality for each relation.

B **GIVEN** a relation from Part 1 of Set A, **CONSTRUCT** a table of ordered pairs from which the graph of the relation could be constructed.

1

A **GIVEN** the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; **IDENTIFY** their $\cap$ by shading on the line.

B **GIVEN** the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; **NAME** the $\cap$ set in terms of the points, line segments, half-lines, or rays.

C **GIVEN** a set of at least four ordered pairs, **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D **GIVEN** an equation from Set B and a value of x, **COMPUTE** the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$ Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$ Note: $\geq$ may be replaced by $=, >, <$, or $\leq$; a, b, c, d, and e are integers.	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$ Note: a, b, and c are integers.

What should you watch for in this lesson? Whenever the author discusses the above topic, perk up! You will be responsible for it "later." When the topic centers around identifying the $\cap$ set of two sets of points such as half-lines, line segments, etc., that is your cue to pay particular attention.

The place of this lesson is shown in red on the Learning Sequence. Lesson III is step 1-A in the Learning Sequence. The author wants you to learn to perform the task of Lesson III so that you will have the necessary skill to learn to perform the task of step 3-A.

Turn the page.

Activity III text material insert.

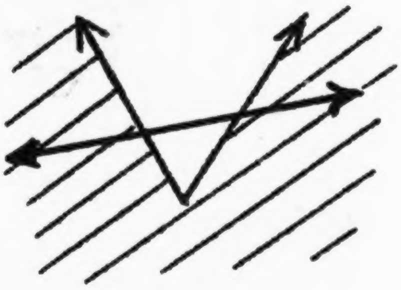
The Objectives of Lesson IV

When you complete Lesson IV, you should be able to perform the following two tasks:

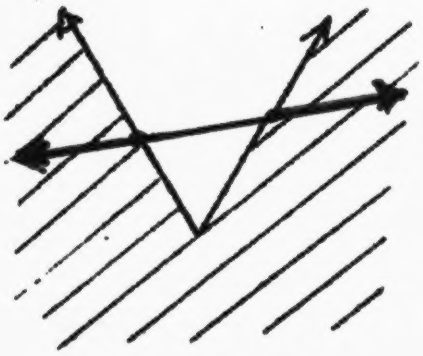
Task 1. GIVEN two graphs plotted on
the same plane;
IDENTIFY, by shading, the $\cap$
of the two graphs.

Example of Task 1:

GIVEN the following two graphs
(one in green and one in black);
IDENTIFY (by coloring in red)
the $\cap$ of the two graphs.



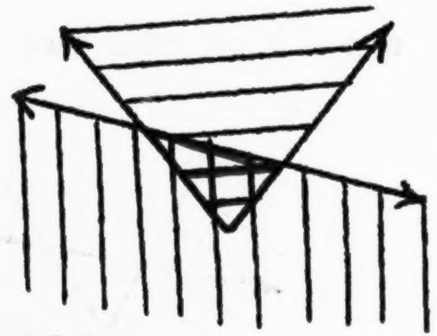
Solution:



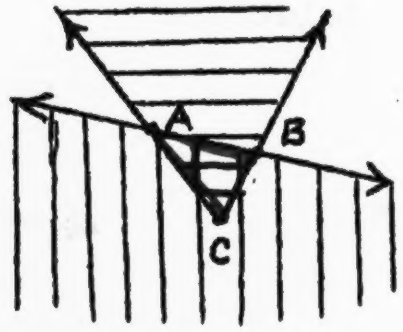
Task 2. GIVEN a figure showing the intersection of two graphs; NAME the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Example of Task 2:

GIVEN the $\cap$ set of the following two graphs; NAME the $\cap$ set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

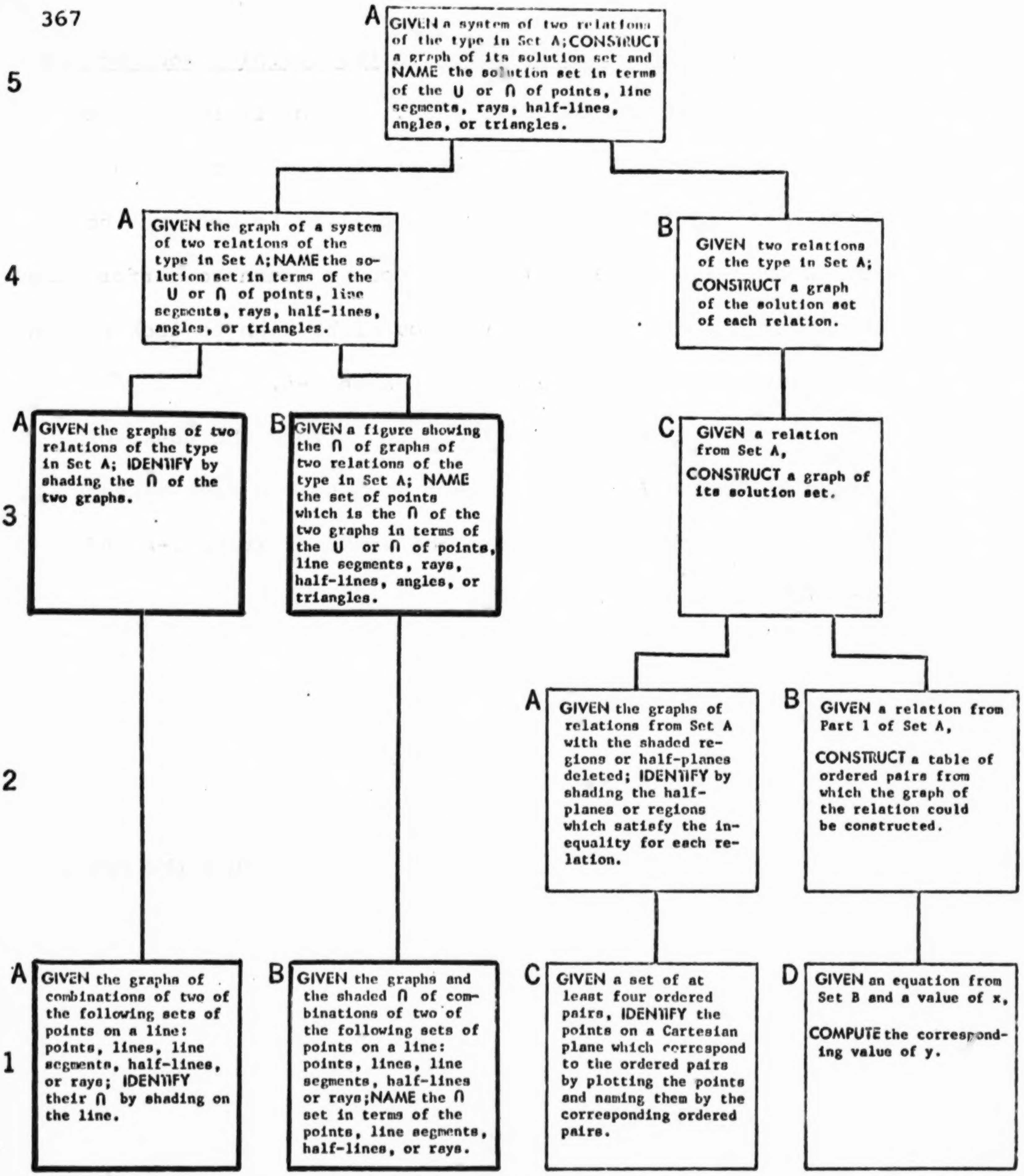


Solution:



$\Delta ABC \cup$ (Interior of ΔABC)

LEARNING SEQUENCE



5

A **GIVEN** a system of two relations of the type in Set A; **CONSTRUCT** a graph of its solution set and **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A **GIVEN** the graph of a system of two relations of the type in Set A; **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B **GIVEN** two relations of the type in Set A; **CONSTRUCT** a graph of the solution set of each relation.

3

A **GIVEN** the graphs of two relations of the type in Set A; **IDENTIFY** by shading the ∩ of the two graphs.

B **GIVEN** a figure showing the ∩ of graphs of two relations of the type in Set A; **NAME** the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C **GIVEN** a relation from Set A, **CONSTRUCT** a graph of its solution set.

2

A **GIVEN** the graphs of relations from Set A with the shaded regions or half-planes deleted; **IDENTIFY** by shading the half-planes or regions which satisfy the inequality for each relation.

B **GIVEN** a relation from Part 1 of Set A, **CONSTRUCT** a table of ordered pairs from which the graph of the relation could be constructed.

1

A **GIVEN** the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; **IDENTIFY** their ∩ by shading on the line.

B **GIVEN** the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; **NAME** the ∩ set in terms of the points, line segments, half-lines, or rays.

C **GIVEN** a set of at least four ordered pairs, **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D **GIVEN** an equation from Set B and a value of x, **COMPUTE** the corresponding value of y.

Set A:		Set B:
Part 1.	$\{(x,y) \mid y \geq \vert x + a \vert ,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$	$y = \vert x + a \vert$ $y = x + b$ $y = c - x$
Part 2.	$\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: ≥ may be replaced by =, >, <, or ≤; a, b, c, d, and e are integers.		

The place of this lesson in the Learning Sequence is shown in red on the OPPOSITE PAGE. Lesson IV is designed to teach you the skills to perform the tasks described in steps 3-A and 3-B. Acquiring the ability to perform the tasks of 3-A and 3-B will enable you to learn to perform the task of step 4-A in Lesson V. You might wish to look now in the green folder at an example of step 4-A.

You will be able to learn to perform the tasks of steps 3-A and 4-A in this lesson because you have already acquired the skill to perform the tasks of steps 1-A and 1-B (shown in blue).

Turn the page.

Activity IV text material insert.

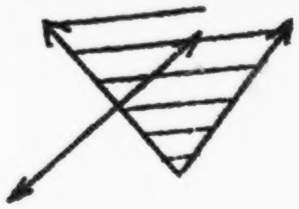
What is Important in Lesson V?

The author has written Lesson V to teach you to perform the following task:

> GIVEN a figure showing a system of two graphs plotted on the same plane; NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

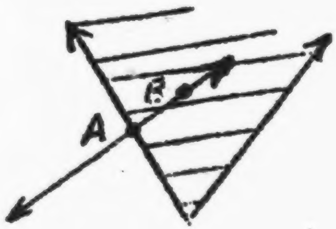
Example of the task:

> GIVEN the following figure of a system of two graphs



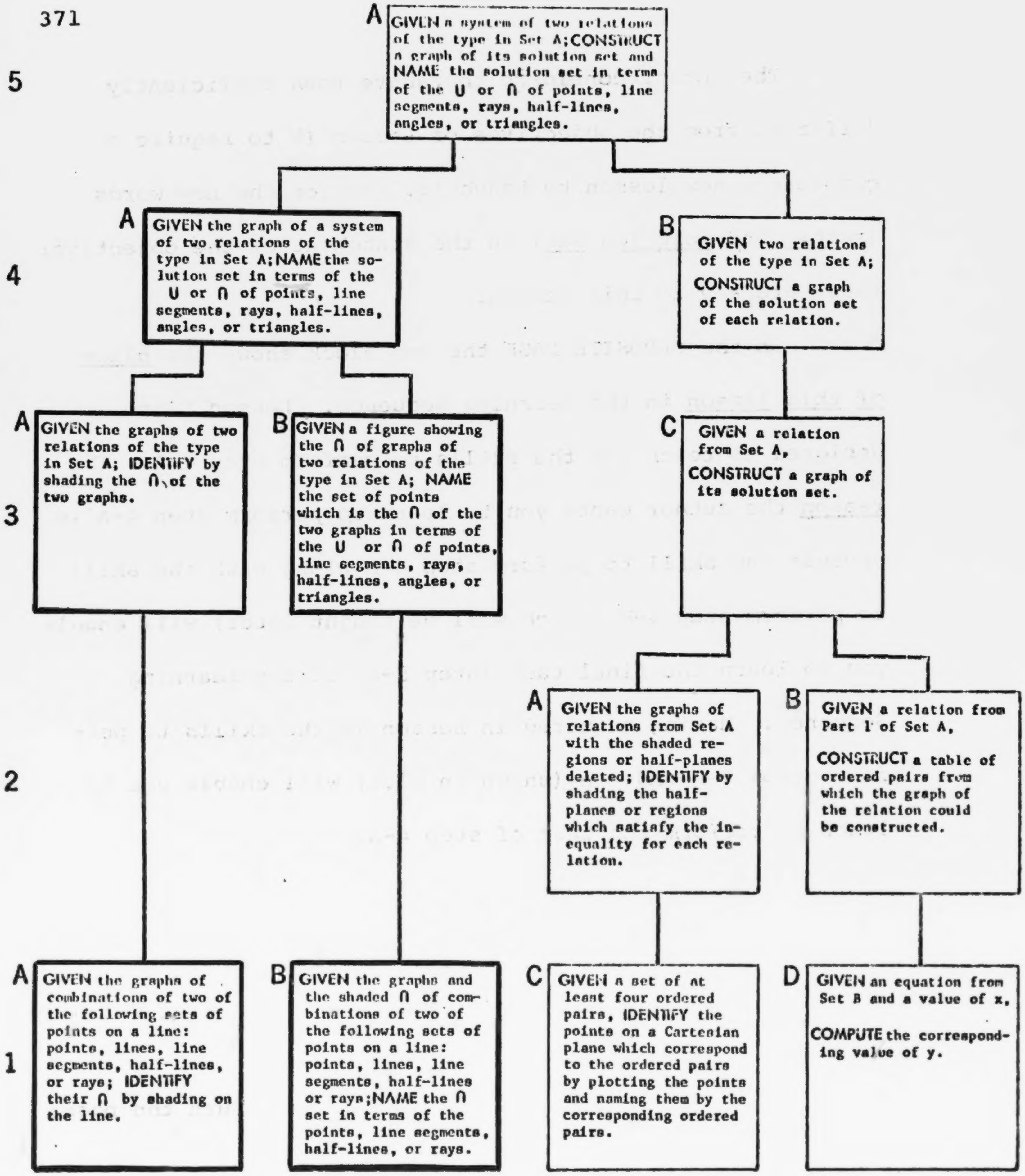
> NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:



$\overrightarrow{AB}$

LEARNING SEQUENCE



5

A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3

A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

B GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2

A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1

A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

B GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x+a \rvert,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x+b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c-x,\ x \text{ reals}\}$ Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid y \geq e,\ x \text{ real}\}$ Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	$y = \lvert x+a \rvert$ $y = x+b$ $y = c-x$ Note: a, b, and c are integers.

The author considers the above task sufficiently different from the objectives of Lesson IV to require a completely new lesson to teach it. Notice the new words (system and solution set) in the statement of the objective; these are key to this lesson.

On the OPPOSITE PAGE the red block shows the place of this lesson in the Learning Sequence. Lesson V is designed to teach you the skills to perform step 4-A. The reason the author wants you to learn to perform step 4-A is because the skill to perform step 4-A along with the skill to perform step 4-B (which will be taught later) will enable you to learn the final task (step 5-A) of the Learning Sequence. Having acquired in Lesson IV the skills to perform steps 3-A and 4-A (shown in blue) will enable you to learn to perform the task of step 4-A.

Turn the page.

Activity V text material insert.

Lesson VI's Objective

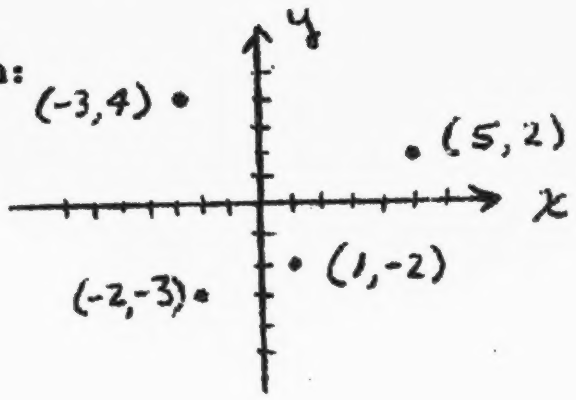
At the end of Lesson VI you should be able to perform the following task:

> **GIVEN** a set of at least four ordered pairs;
> **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

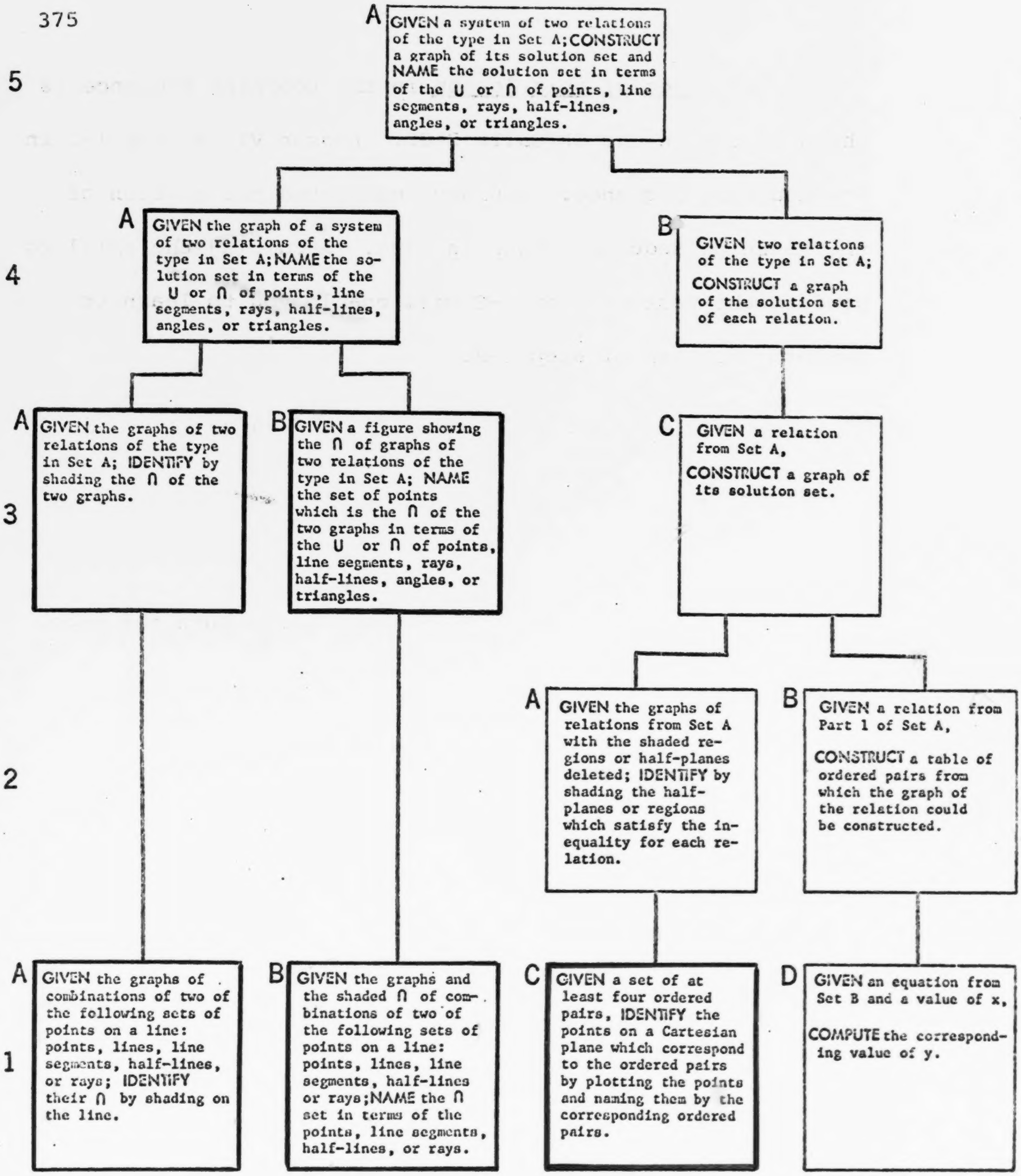
Example of the task:

> **GIVEN** the set of ordered pairs:
>
> $\{(-2, -3), (1, -2), (5, 2), (-3, 4)\}$
>
> **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting and naming them by the corresponding ordered pairs.

Solution:



LEARNING SEQUENCE



5

A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3

A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

B GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2

A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1

A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

B GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x+a \rvert,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x+b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c-x,\ x \text{ reals}\}$	$y = \lvert x+a \rvert$ $y = x+b$ $y = c-x$
Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

The place of this lesson in the Learning Sequence is shown in red on the OPPOSITE PAGE. Lesson VI is step 1-C in the Learning Sequence. You have completed the portion of the Learning Sequence shown in blue. Acquiring the skill to perform the task of step 1-C will enable you to learn to perform the task of step 2-A.

Turn the page.

Activity VI _text_ _material_ _insert_.

The Objectives of Lesson VII

Lesson VII is divided into two parts: Lesson VII-a and Lesson VII-b.

When you complete Lesson VII-a, you should be able to perform the following task:

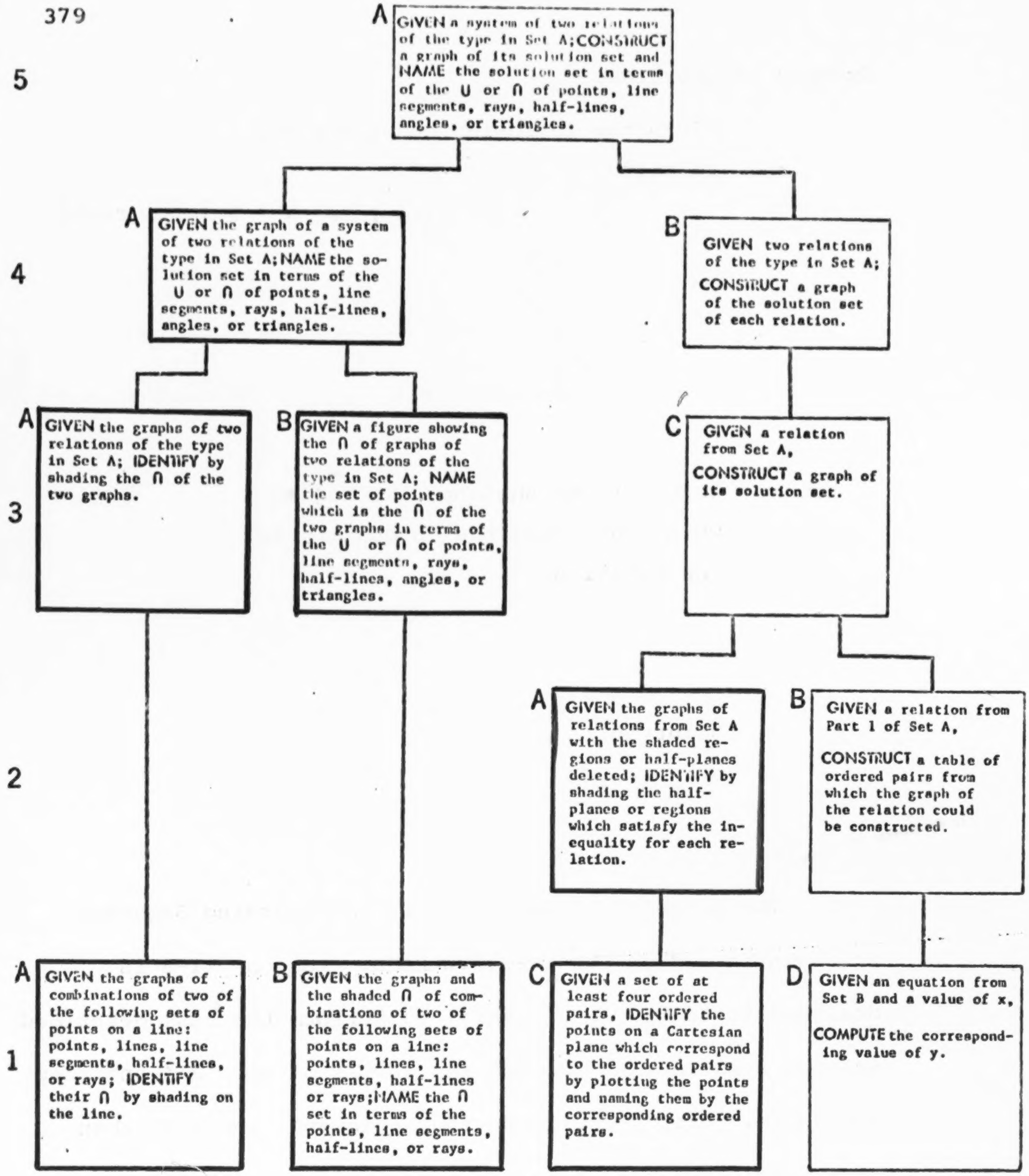
> GIVEN the graphs of relations from Set A (see below) with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

The Set A referred to in the above statement of the objective of Lesson VII-a is:

$$\{(x, y) \mid y \geq |x + a|, \ x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b, \ x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x, \ x \text{ real}\}$$
$$\{(x, y) \mid y \geq d, \ x \text{ real}\}$$
$$\{(x, y) \mid x \geq e, \ x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$;
a, b, c, d, and e are integers.

LEARNING SEQUENCE



5

A **GIVEN** a system of two relations of the type in Set A; **CONSTRUCT** a graph of its solution set and **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A **GIVEN** the graph of a system of two relations of the type in Set A; **NAME** the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B **GIVEN** two relations of the type in Set A; **CONSTRUCT** a graph of the solution set of each relation.

3

A **GIVEN** the graphs of two relations of the type in Set A; **IDENTIFY** by shading the ∩ of the two graphs.

B **GIVEN** a figure showing the ∩ of graphs of two relations of the type in Set A; **NAME** the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C **GIVEN** a relation from Set A, **CONSTRUCT** a graph of its solution set.

2

A **GIVEN** the graphs of relations from Set A with the shaded regions or half-planes deleted; **IDENTIFY** by shading the half-planes or regions which satisfy the inequality for each relation.

B **GIVEN** a relation from Part 1 of Set A, **CONSTRUCT** a table of ordered pairs from which the graph of the relation could be constructed.

1

A **GIVEN** the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; **IDENTIFY** their ∩ by shading on the line.

B **GIVEN** the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; **NAME** the ∩ set in terms of the points, line segments, half-lines, or rays.

C **GIVEN** a set of at least four ordered pairs, **IDENTIFY** the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

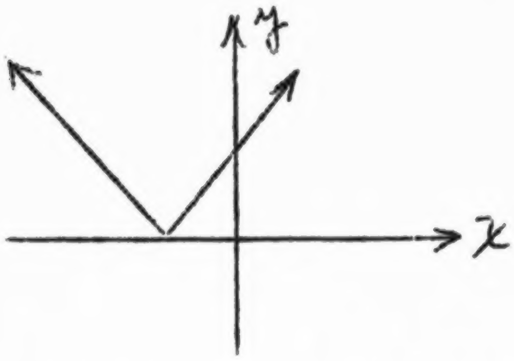
D **GIVEN** an equation from Set B and a value of x, **COMPUTE** the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert ,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$ Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$ Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$ Note: a, b, and c are integers.

Example of the task:

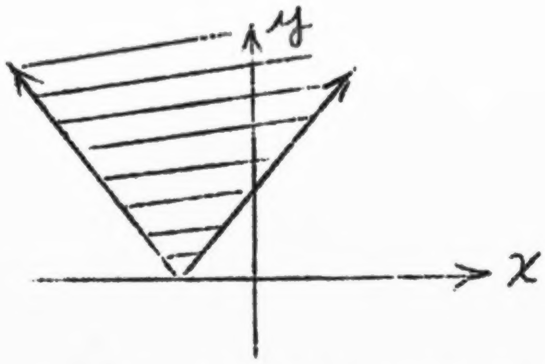
GIVEN the following curve for the relation

$\left\{(x, y) \mid y \geq |x + 2|, x \text{ real}\right\}$



IDENTIFY, by shading, the region which satisfies the inequality for the relation.

Solution:



The place of Lesson VII-a in the Learning Sequence is shown in red on the OPPOSITE PAGE. Lesson VII-a is designed to teach you the skill to perform the task described in step 2-A. Having learned the skills of step 2-A and step 2-B (note: step 2-B will be taught later), you will then have sufficient skills to learn to perform the task of step 3-C. It will benefit you if you will refer to your GREEN reference folder to see an example of the task in step 3-C.

Now you know why the author wants you to learn the skill of Lesson VII-a. The reason you will be able to learn this skill is because you have already learned the skill of step 1-C shown in blue on the Learning Sequence.

You can see by the number of the blue blocks on the Learning Sequence that you have learned several skills thus far. Each skill you have learned has enabled you to progress to another step in the Learning Sequence.

Turn the page.

Activity VIIa text material insert.

The Objectives of Lesson VII Continued

When you have completed Lesson VII-b, you should be able to perform the following three tasks:

Task 1: GIVEN an equation from
Set B and a value of x,
COMPUTE the corresponding
value of y.

The Set B referred to in the above statement of Task 1 is:

$y = |x + a|$
$y = x + b$
$y = c - x$
Note: a, b, and c are integers

Example of Task 1:

GIVEN the equation
$y = |x - 5|$
with $y = -4$
COMPUTE the corresponding
value of y.

Solution: $y = |-4 - 5| = |-9| = 9$

Task 2: GIVEN a relation from the set of relations shown below, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

The set of relations referred to in the above statement of Task 2 is:

$$\left\{(x, y) \mid y \geq |x + a| \text{ , } x \text{ real}\right\}$$
$$\left\{(x, y) \mid y \geq x + b \text{ , } x \text{ real}\right\}$$
$$\left\{(x, y) \mid y \geq c - x \text{ , } x \text{ real}\right\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$;
a, b, c, d, and e are integers.

Example of Task 2:

GIVEN the relation

$$\left\{(x, y) \mid y \leq 3 - x \text{ , } x \text{ real}\right\}$$

CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

Solution:

x	-1	1	3
y	4	2	0

Task 3: GIVEN a relation from
Set A,
CONSTRUCT a graph of
its solution set.

The Set A referred to in the above statement of Task 3 is:

$$\{(x, y) \mid y \geq |x + a| , x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b , x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x , x \text{ real}\}$$
$$\{(x, y) \mid y \geq d , x \text{ real}\}$$
$$\{(x, y) \mid x \geq e , x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$;
a, b, c, d, and e are integers.

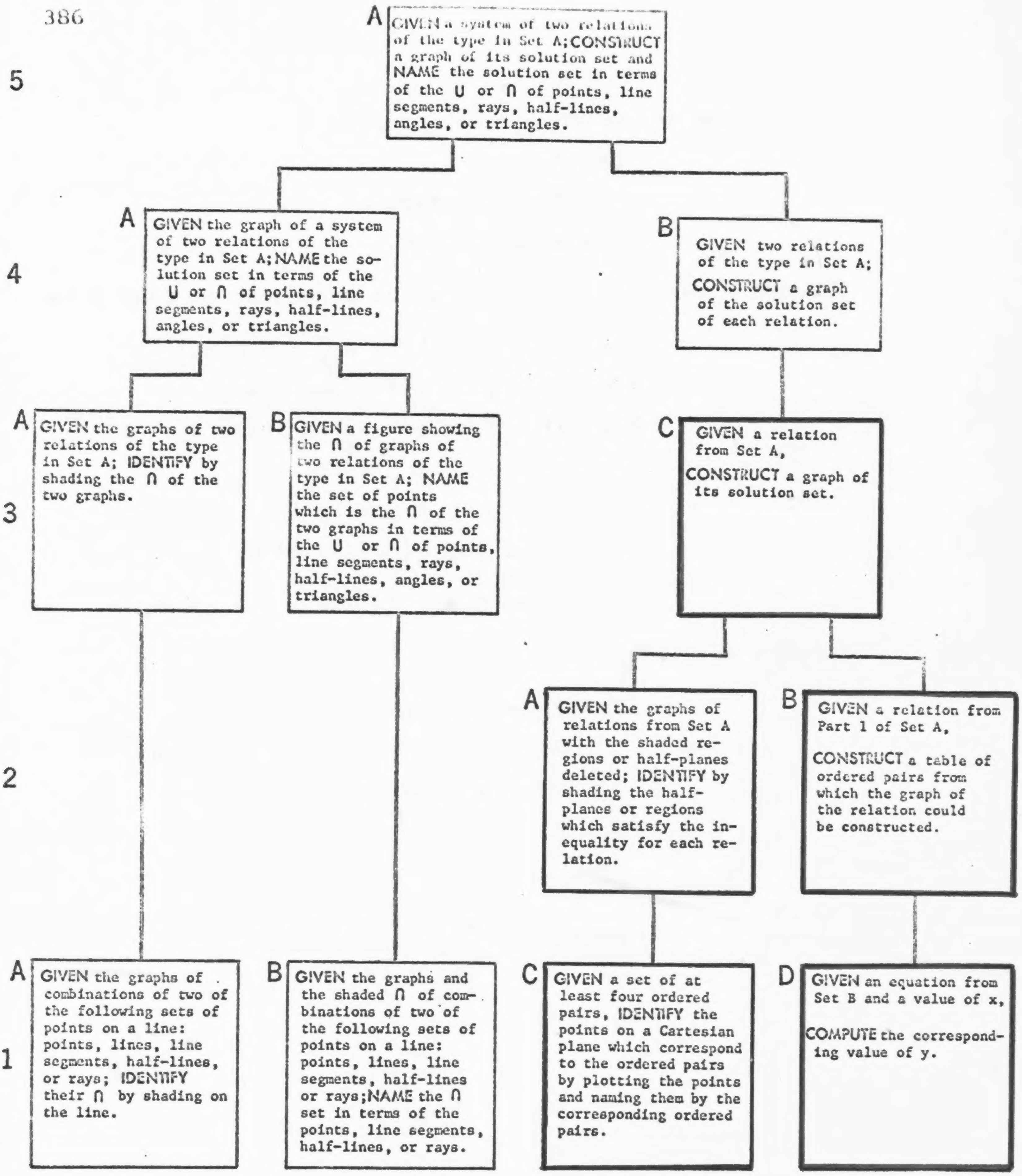
Example of Task 3:

GIVEN the relation
$$\{(x, y) \mid y \geq |x + 2| , x \text{ real}\}$$
CONSTRUCT a graph of its solution set.

Solution:

x	-6	-4	-2	0	2
y	4	2	0	2	4

LEARNING SEQUENCE



A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

B GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

B GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a\rvert,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x + b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c - x,\ x \text{ reals}\}$	$y = \lvert x + a\rvert$ $y = x + b$ $y = c - x$
Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

The place of Lesson VII-b is shown on the OPPOSITE PAGE by the three red blocks. The lesson begins with the task of step 1-D and progresses to step 2-B and then to step 3-C. Having already acquired the skill to perform the task of step 2-A (shown in blue), acquiring the skill to perform the task of step 2-B will equip you to learn to perform the task of step 3-C.

Why does the author want you to acquire the skill for step 3-C ? He considers the skill of step 3-C will enable you to learn to perform the task of step 4-B. Lesson VIII-a will be concerned with step 4-B. Refer to the example of step 4-B in the green folder to obtain an idea of what type of task Lesson VII-b leads into.

Turn the page.

Activity VIIb text material insert.

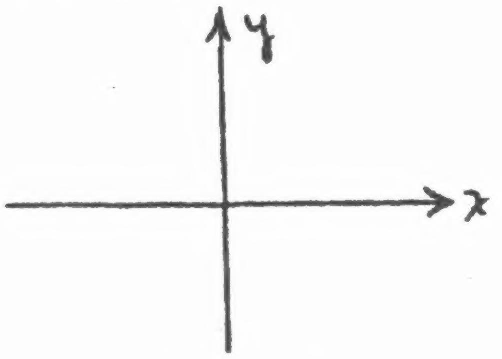
The Objectives of Lesson VIII

Lesson VIII has two parts: VIII-a and VIII-b. The objective of Lesson VIII-a is given here while the objective of Lesson VIII-b is stated later. The objective of Lesson VIII-a is to teach you to perform the following task:

> GIVEN two relations of the type in Set A;
> CONSTRUCT a graph of the solution set of each relation on the same Cartesian plane.

Example of the task:

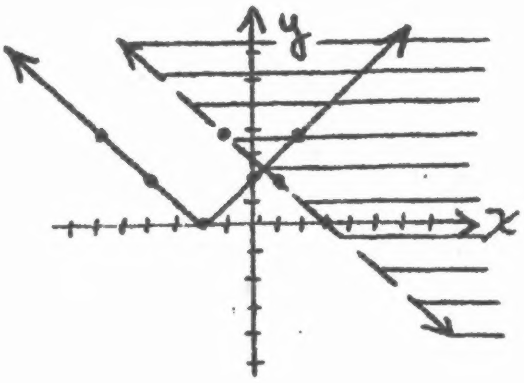
> GIVEN the following two relations
>
> $$\{(x, y) \mid y > 3 - x \text{ , } x \text{ real}\}$$
>
> and $$\{(x, y) \mid y = |x + 2| \text{ , } x \text{ real}\}$$
>
> CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



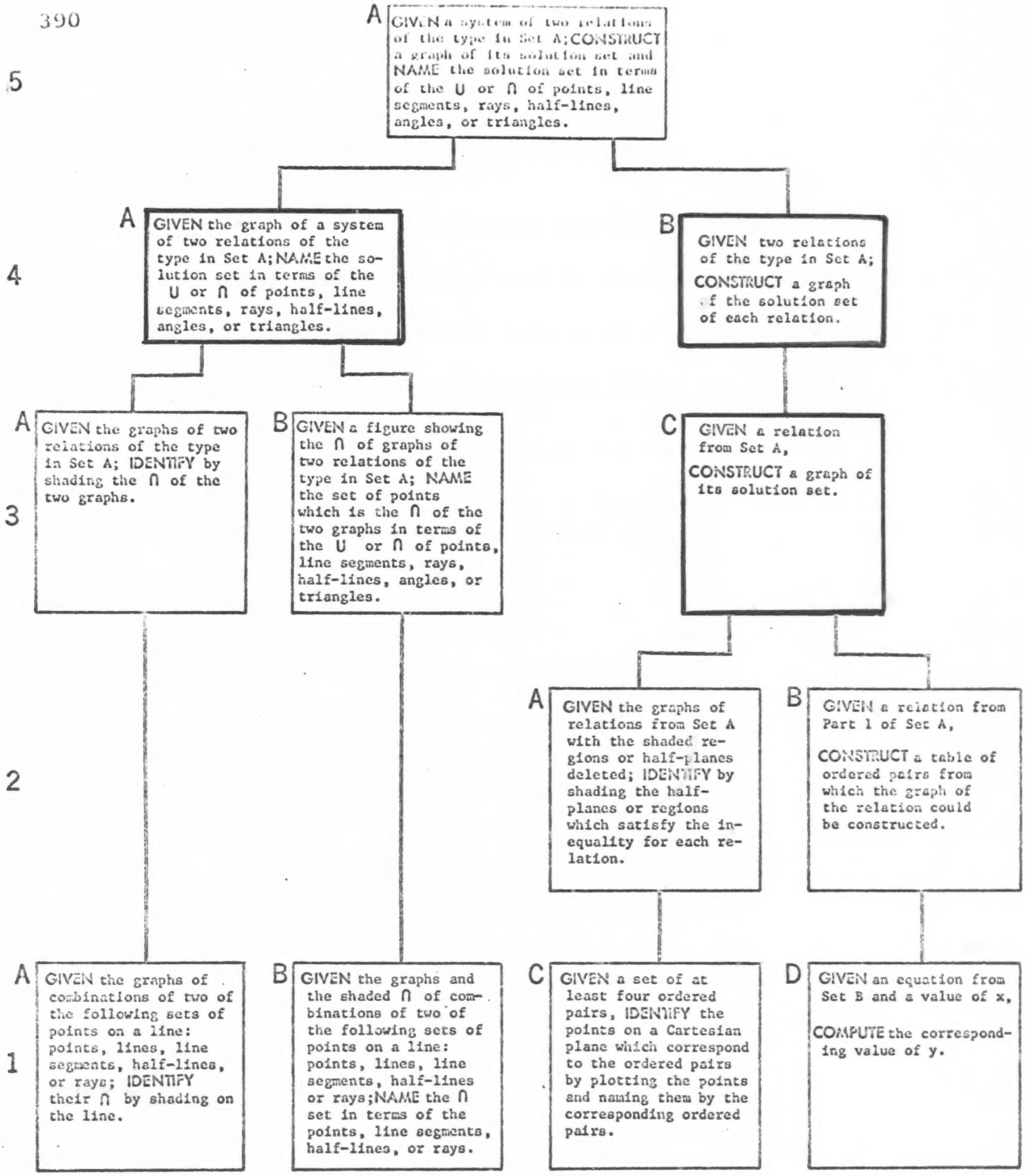
Solution:

x	-1	1	3
y	4	2	0

x	-6	-4	-2	0	2
y	4	2	0	2	4



LEARNING SEQUENCE



5

A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

4

A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3

A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the ∩ of the two graphs.

B GIVEN a figure showing the ∩ of graphs of two relations of the type in Set A; NAME the set of points which is the ∩ of the two graphs in terms of the ∪ or ∩ of points, line segments, rays, half-lines, angles, or triangles.

C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2

A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1

A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their ∩ by shading on the line.

B GIVEN the graphs and the shaded ∩ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the ∩ set in terms of the points, line segments, half-lines, or rays.

C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x + a \rvert , x \text{ real}\}$ $\{(x,y) \mid y \geq x + b, x \text{ real}\}$ $\{(x,y) \mid y \geq c - x, x \text{ reals}\}$	$y = \lvert x + a \rvert$ $y = x + b$ $y = c - x$
Part 2. $\{(x,y) \mid y \geq d, x \text{ real}\}$ $\{(x,y) \mid x \geq e, x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

The place of Lesson VIII-a in the Learning Sequence is shown in red on the OPPOSITE PAGE. You have previously acquired in Lesson V the skill to perform the task of step 4-A. At the end of Lesson VIII-a you will have acquired the skill of the task described in step 4-B. With these two skills you will then be ready to learn to perform the terminal task (step V-A) of the Learning Sequence. Lesson VIII-b will teach you to perform the task of V-A. Refer to the example of step V-A in the green folder now to see the type of task Lesson VIII-a leads into.

Turn the page.

Activity VIIIa _text material insert_.

The Objective of Lesson VIII Continued

The objective of Lesson VIII-b is essentially the objective of the entire Learning Sequence we have been following. The objective of Lesson VIII-b is considered by the author to be the purpose for which all the previous lessons have been taught. When you have completed Lesson VIII-b, you should be able to perform this terminal task:

> GIVEN a system of two relations
> of the type in Set A;
> CONSTRUCT a graph of its solution set and
> NAME the solution set in terms of the
> $\cup$ or $\cap$ of points, line segments, rays,
> half-lines, angles, or triangles.

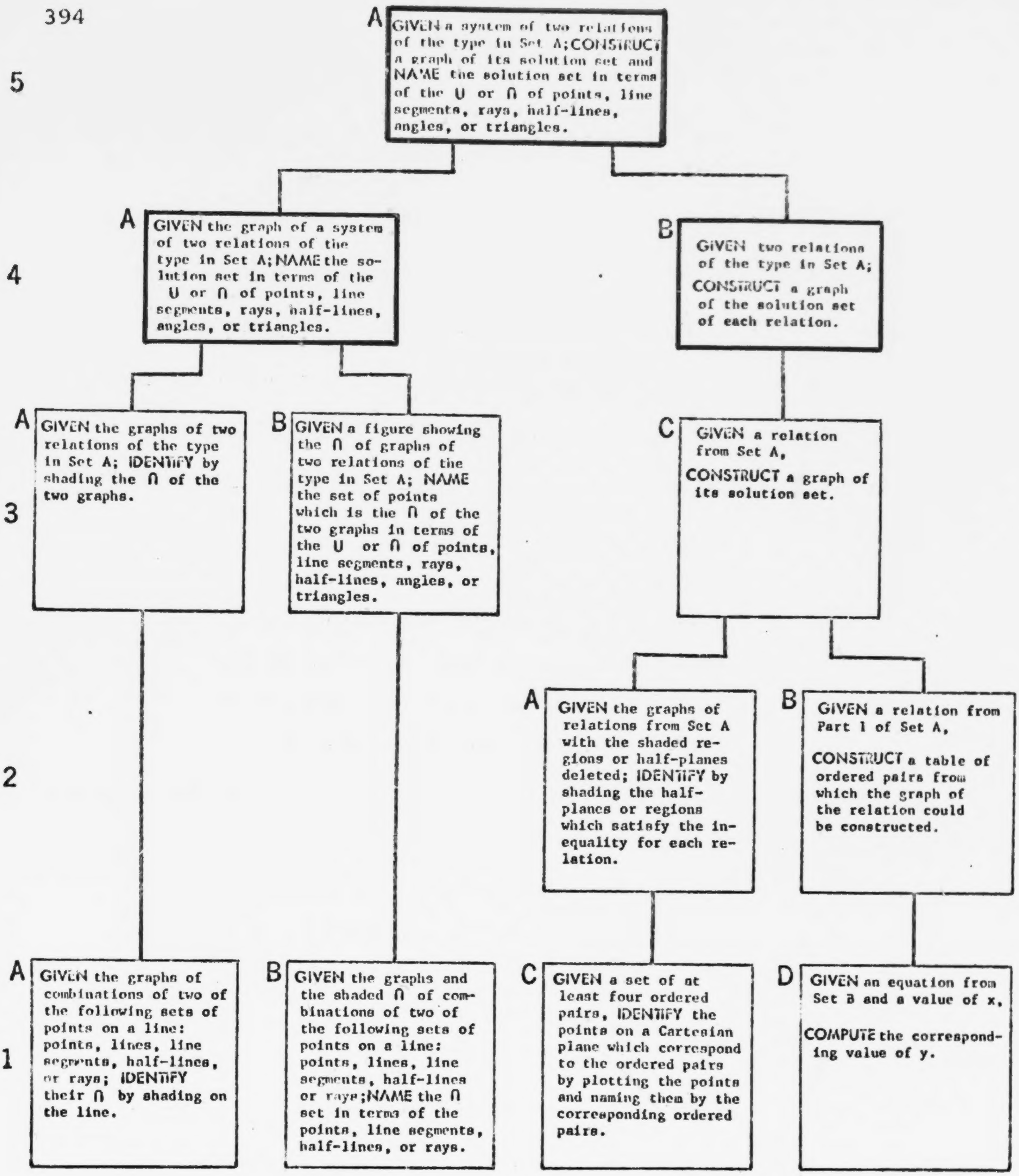
The Set A referred to in the above statement of the terminal task is:

$$\{(x, y) \mid y \geq |x + a| , x \text{ real}\}$$
$$\{(x, y) \mid y \geq x + b , x \text{ real}\}$$
$$\{(x, y) \mid y \geq c - x , x \text{ real}\}$$
$$\{(x, y) \mid y \geq d , x \text{ real}\}$$
$$\{(x, y) \mid x \geq e , x \text{ real}\}$$

Note: $\geq$ may be replaced by $=$, $>$, $<$,
or $\leq$;
a, b, c, d, and e are integers.

LEARNING SEQUENCE

394



5

A GIVEN a system of two relations of the type in Set A; CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

4

A GIVEN the graph of a system of two relations of the type in Set A; NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

B GIVEN two relations of the type in Set A; CONSTRUCT a graph of the solution set of each relation.

3

A GIVEN the graphs of two relations of the type in Set A; IDENTIFY by shading the $\cap$ of the two graphs.

B GIVEN a figure showing the $\cap$ of graphs of two relations of the type in Set A; NAME the set of points which is the $\cap$ of the two graphs in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.

C GIVEN a relation from Set A, CONSTRUCT a graph of its solution set.

2

A GIVEN the graphs of relations from Set A with the shaded regions or half-planes deleted; IDENTIFY by shading the half-planes or regions which satisfy the inequality for each relation.

B GIVEN a relation from Part 1 of Set A, CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

1

A GIVEN the graphs of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines, or rays; IDENTIFY their $\cap$ by shading on the line.

B GIVEN the graphs and the shaded $\cap$ of combinations of two of the following sets of points on a line: points, lines, line segments, half-lines or rays; NAME the $\cap$ set in terms of the points, line segments, half-lines, or rays.

C GIVEN a set of at least four ordered pairs, IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.

D GIVEN an equation from Set B and a value of x, COMPUTE the corresponding value of y.

Set A:	Set B:
Part 1. $\{(x,y) \mid y \geq \lvert x+a\rvert ,\ x \text{ real}\}$ $\{(x,y) \mid y \geq x+b,\ x \text{ real}\}$ $\{(x,y) \mid y \geq c-x,\ x \text{ reals}\}$	$y = \lvert x+a\rvert$ $y = x+b$ $y = c-x$
Part 2. $\{(x,y) \mid y \geq d,\ x \text{ real}\}$ $\{(x,y) \mid x \geq e,\ x \text{ real}\}$	Note: a, b, and c are integers.
Note: $\geq$ may be replaced by $=$, $>$, $<$, or $\leq$; a, b, c, d, and e are integers.	

Example of the terminal task:

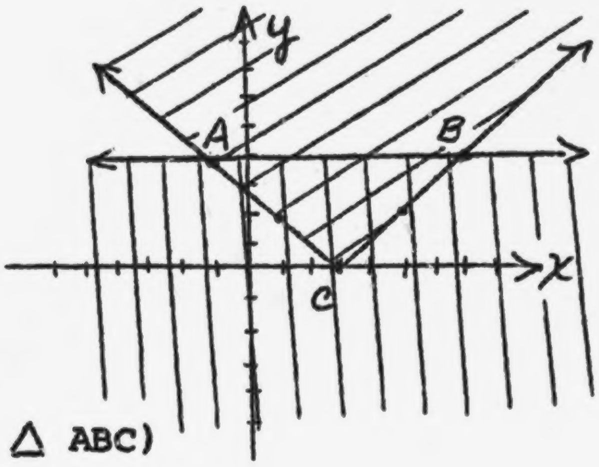
GIVEN the following system of two relations

$$\begin{cases} \{(x, y) \mid y \geq |x - 3|, \; x \text{ real}\} \\ \{(x, y) \mid y \leq 4, \; x \text{ real}\} \end{cases}$$

CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cap$ or $\cup$ of points, line segments, rays, half-lines, angles, or triangles.

Solution:

x	-1	1	3	5	7
y	4	2	0	2	4



$\triangle$ ABC $\cup$ (Interior of $\triangle$ ABC)

The place of Lesson VIII-b is shown in red in the Learning Sequence on the OPPOSITE PAGE. The balance of the Learning Sequence shown in blue has been to prepare you to learn to perform the terminal task of step V-A. YOUR LEARNING TO PERFORM THIS TASK IS WHAT THE AUTHOR CONSIDERS TO BE THE OBJECTIVE OF THE ENTIRE LEARNING SEQUENCE.

Turn the page.

Activity VIIIb text material insert.

APPENDIX E

"CHECK-UP QUIZZES"

"Check-up" Quizzes
for
Treatments 2, 3, and 4
Treatment 1

Lesson IV--"Check-up" Questions

Name ______________________________

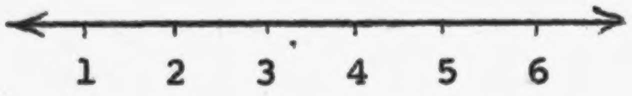
Time: 10:00 _____; 11:00 _____; 12:00 _____; 1:00 _____ .

INSTRUCTIONS:

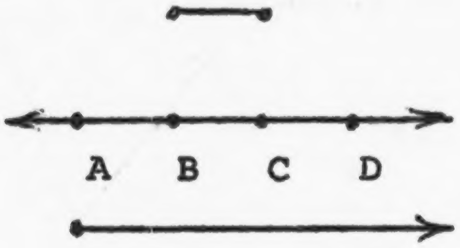
A. Circle the number of the problems which are examples of the objectives of this lesson.

B. Work only the problems you circled.

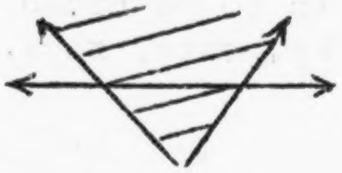
1. Graph the set $[2, 4] \cap (3, 5)$



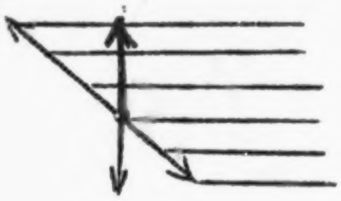
2. One set of points on the line $\overleftrightarrow{AD}$ is represented above $\overleftrightarrow{AD}$ while a second set of points on $\overleftrightarrow{AD}$ is represented below $\overrightarrow{AD}$. IDENTIFY the $\cap$ of the two sets of points in each of the following by shading line $\overleftrightarrow{AD}$ in red over the appropriate portion.



3. IDENTIFY, by coloring in red, the $\cap$ of the two following graphs:



4. NAME the $\cap$ set shown in red of the two following graphs.



Lesson V--"Check-up" Questions

Name ______________________________

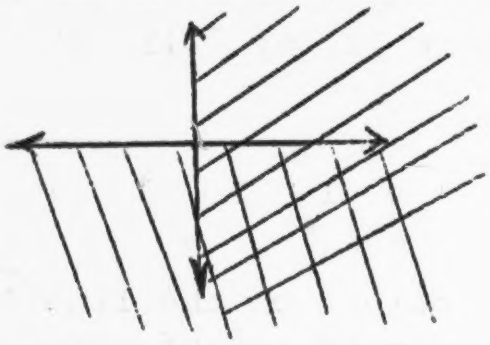
Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

INSTRUCTIONS:

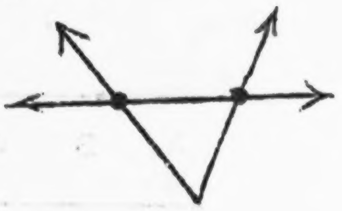
A. Circle the number of the problems which are examples of the objectives of this lesson.

B. Work only the problems you circled.

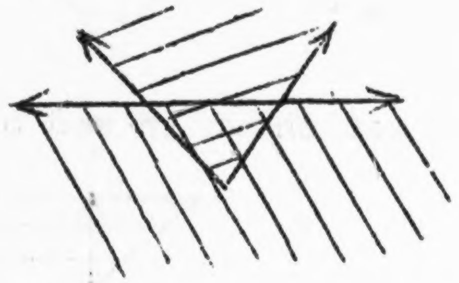
1. IDENTIFY, by coloring in red, the $\cap$ of the two following graphs:



2. NAME the $\cap$ set, shown in red, of the two following graphs. The $\cap$ set is to be named in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles:



3. NAME the solution set of the system of two graphs. The solution set is to be named in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.



Lesson VI--"Check-up" Questions

Name ____________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

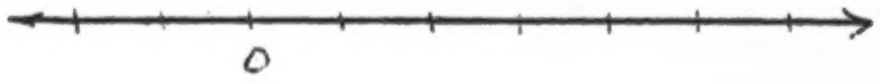
INSTRUCTIONS:

A. Circle the number of the problems which are examples of the objectives of this lesson.
B. Work only the problems you circled.

1. GIVEN the set of integers,

$\{-2, 0, 3, 5, 6\}$

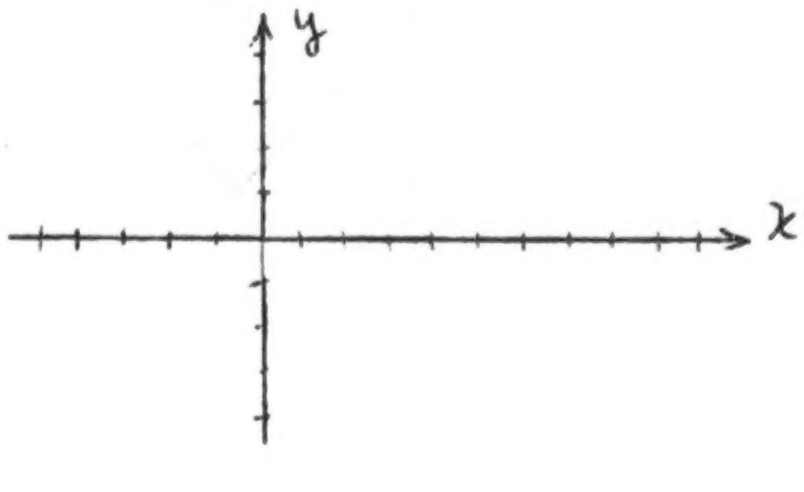
IDENTIFY their corresponding points on the following number lines by plotting the points and naming them by the corresponding integers.



2. GIVEN the set of ordered pairs

$\{(-3, 6), (-1, 4), (0, 3), (3, 0), (5, 2), (7, 4)\}$

IDENTIFY the points on a Cartesian plane which correspond to the ordered pairs by plotting the points and naming them by the corresponding ordered pairs.



Lesson VII-a--"Check-up" Questions

Name ______________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

INSTRUCTIONS:

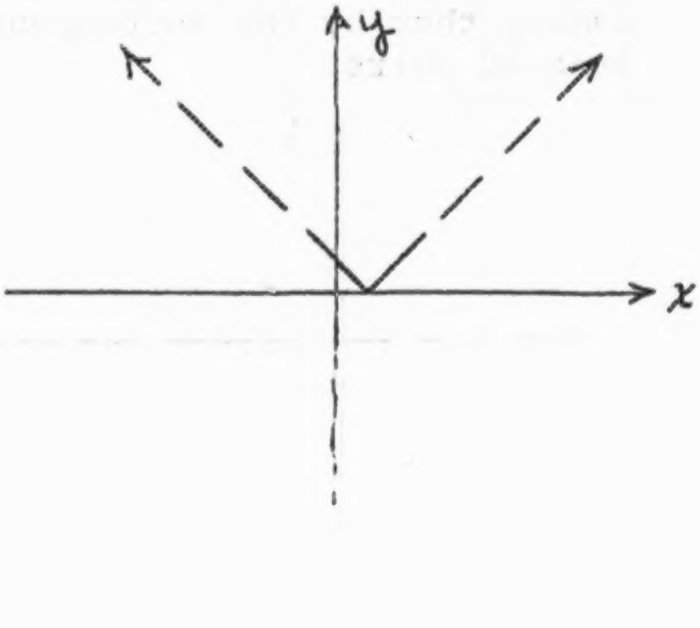
A. Circle the number of the problems which are examples of the objectives of this lesson.
B. Work only the problems you circled.

1. CONSTRUCT the graph of the relation $\{(x, y) \mid y = x + 3 \text{ , } x \text{ real}\}$.

2. GIVEN the value of x to be -3 , COMPUTE the value of

$$|x - 5|$$

3. GIVEN the relation $\{(x, y) \mid y > |x - 1| \text{, } x \text{ real}\}$ and its graph with the shaded region deleted; IDENTIFY by shading the region which satisfies $y > |x - 1|$.



Lesson VII b--"Check-up" Questions

Name ______________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

INSTRUCTIONS:

A. Circle the number of the problems which are examples of the objectives of this lesson.

B. Work only the problems you circled.

1. GIVEN the relation

$$\{(x, y) \mid y > |x - 3|, x \text{ real}\}$$

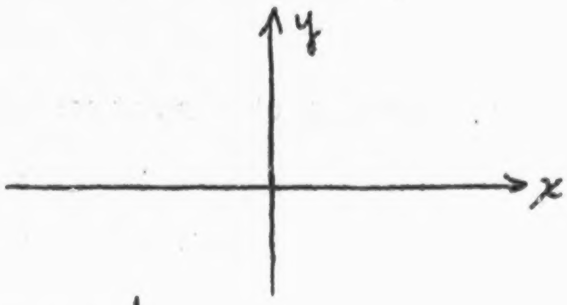
CONSTRUCT a table of ordered pairs from which the graph of the relation could be constructed.

x	
y	

2. GIVEN the relation

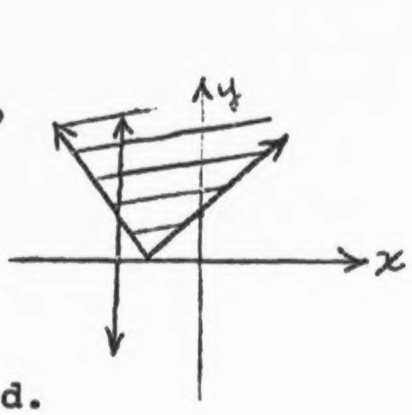
$$\{(x, y) \mid y < x + 2, x \text{ real}\}$$

CONSTRUCT a graph of its solution set.



3. GIVEN the equation $y = |x + 3|$
COMPUTE the corresponding value of y.

4. GIVEN the following graphs of two different relations:



IDENTIFY their $\cap$ by shading in red.

Lesson VIII--"Check-up" Questions

Name ________________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____.

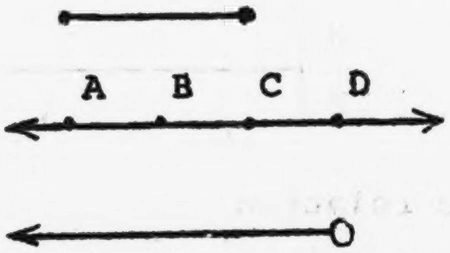
THESE INSTRUCTIONS ARE DIFFERENT:

A. Draw a rectangle around the number of the problem which is an example of the objective of Lesson VIII-a.

B. Circle the number of the problem which is an example of the objective of Lesson VIII-b.

C. Work all problems.

1. GIVEN the line $\overleftrightarrow{AD}$. The graph of one set of points ($\overline{AC}$) on $\overleftrightarrow{AD}$ is shown above $\overleftrightarrow{AD}$ and the graph of a second set of points (half-line DA) on $\overleftrightarrow{AD}$ is shown below $\overleftrightarrow{AD}$.

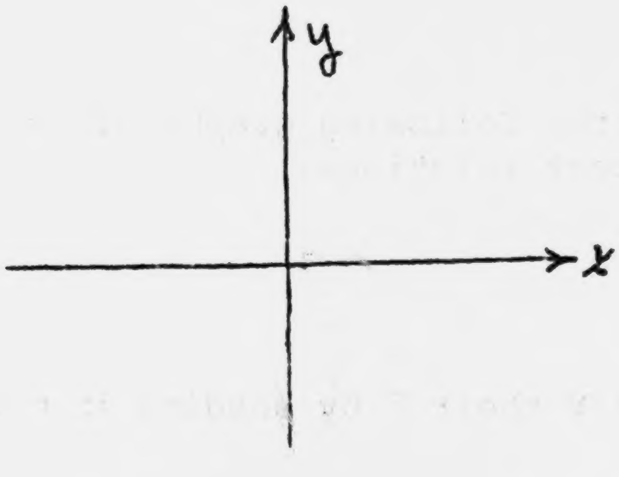


IDENTIFY the intersection of $\overline{AC}$ and the half-line DA by shading in red on line $\overleftrightarrow{AD}$.

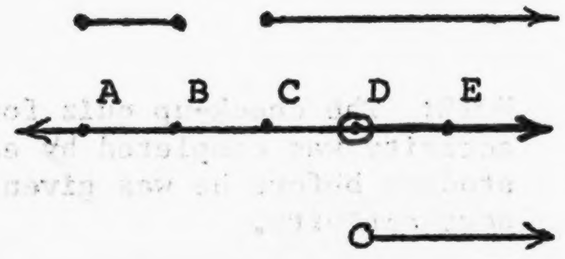
2. GIVEN the system of two relations

$$\left\{\begin{array}{l} \{(x, y) \mid y > |x - 1|, \ x \text{ real}\} \\ \{(x, y) \mid y < 3, \ x \text{ real}\} \end{array}\right.$$

CONSTRUCT a graph of its solution set and NAME the solution set in terms of the $\cup$ or $\cap$ of points, line segments, rays, half-lines, angles, or triangles.



3. **GIVEN** the line $\overleftrightarrow{AE}$. The graph of the set of points $\overline{AB} \cup \overrightarrow{CE}$ on $\overleftrightarrow{AE}$ is shown above $\overleftrightarrow{AE}$ and the set of points half-line DE on $\overleftrightarrow{AE}$ is shown below $\overleftrightarrow{AE}$. The $\cap$ of $\overline{AB} \cup \overrightarrow{CE}$ and half-line DE is shown in red on $\overleftrightarrow{AE}$.



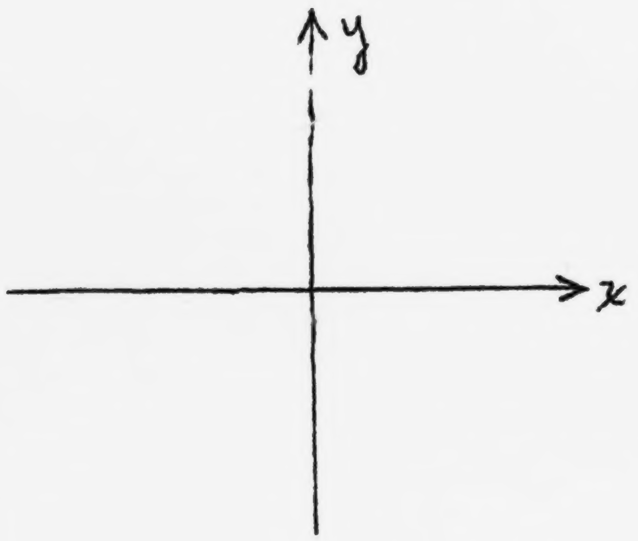
NAME the $\cap$ set in terms of points, line segments, half-lines, or rays.

4. **GIVEN** the following two relations

$$\{(x, y) \mid y \geq |x + 1| \text{ , x real}\}$$

and $\{(x, y) \mid y = 2 \text{ , x real}\}$

CONSTRUCT a graph of the solution set of each relation on the Cartesian plane provided below:



NOTE: The check-up quiz for each activity was completed by each student before he was given the next activity.

The "Check-up" Quizzes for the students in Treatment 1 were the same as the quizzes for the other three treatment groups with the exception that no instructions were stated on the quizzes.

APPENDIX F

PERFORMANCE TEST

Performance Test

Version 1

Name ____________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

Check the number of extra sessions you have attended. Do not count extra sessions which were because of absences.

1 ____; 2 ____; 3 ____; 4 ____; 5 ____; 6 ____ .

The last day I worked on the self-instructional material (including "check-up") was : Wednesday ____ .
Friday ____ .

Name ____________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

Write your reaction to the style and format of the self-instructional material you have used during the last three weeks.

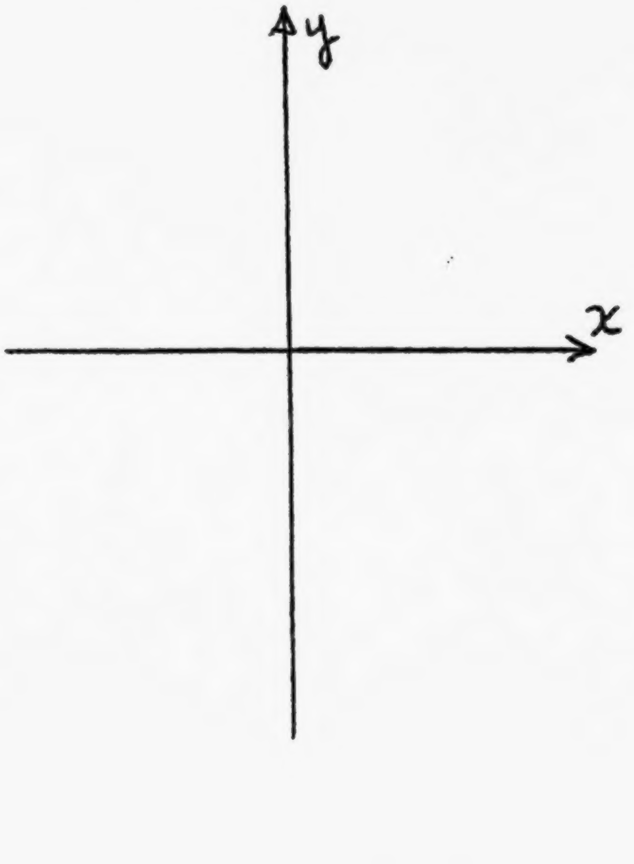
Name ______________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

There are 7 problems in the test. Each problem is valued at 14 points on the test. The instructions for all 7 problems are

> GIVEN the system of two relations
> in each of the following problems,
> CONSTRUCT a graph of the solution set
> of each system and
> NAME the solution set in terms of
> the $\cup$ or $\cap$ of points, line segments,
> rays, half-lines, angles, or triangles.

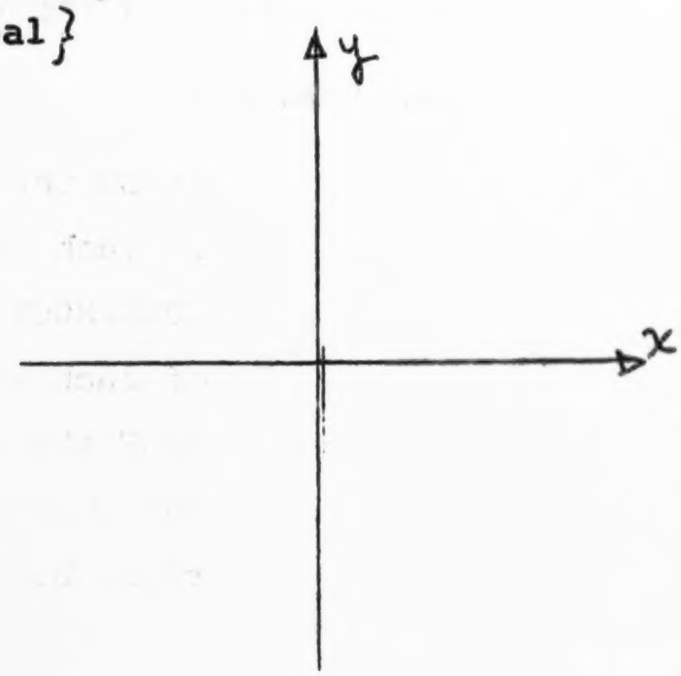
$$\begin{cases} \{(x, y) \mid y \le |x - 1| \text{ , } x \text{ real}\} \\ \{(x, y) \mid y = 2 \text{ , } x \text{ real}\} \end{cases}$$



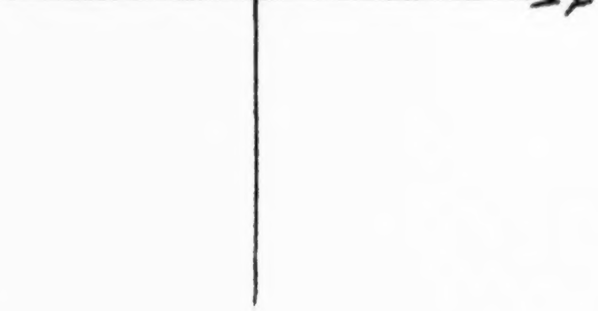
Name ____________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

$$\begin{cases} \{(x, y) \mid y = |x + 4|,\ x \text{ real}\} \\ \{(x, y) \mid y \leq x - 4,\ x \text{ real}\} \end{cases}$$



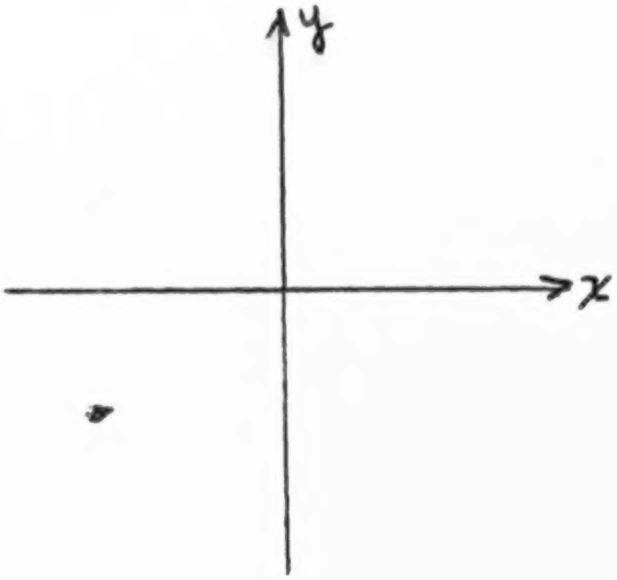
$$\begin{cases} \{(x, y) \mid y < 3 - x,\ x \text{ real}\} \\ \{(x, y) \mid y > x + 2,\ x \text{ real}\} \end{cases}$$



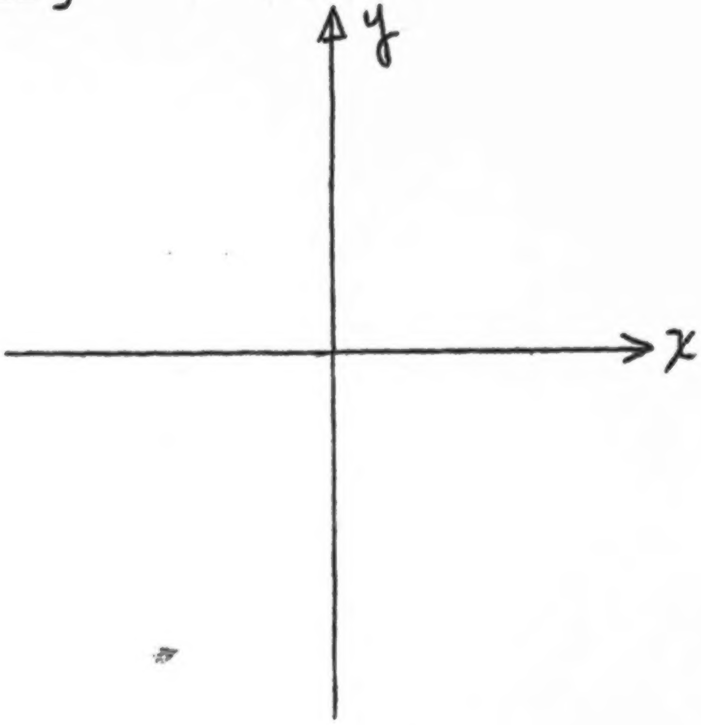
Name ______________________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____.

$$\begin{cases} \{(x, y) \mid y \geq |x - 4| , x \text{ real}\} \\ \{(x, y) \mid y \leq 5 , x \text{ real}\} \end{cases}$$



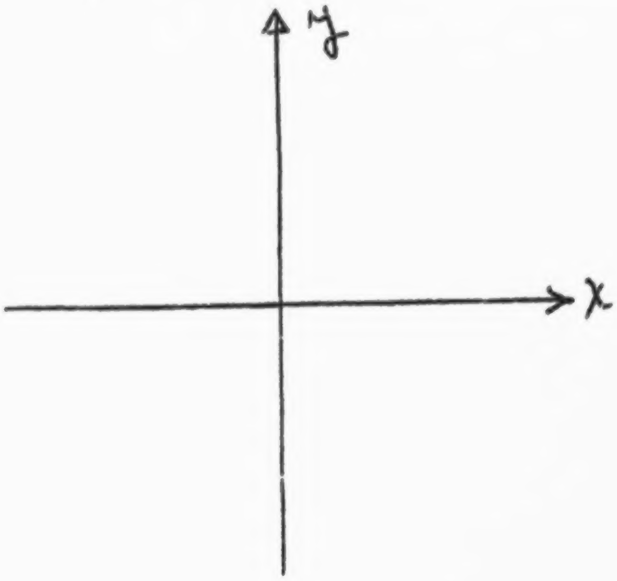
$$\begin{cases} \{(x, y) \mid y = 1 - x , x \text{ real}\} \\ \{(x, y) \mid x < 4 , x \text{ real}\} \end{cases}$$



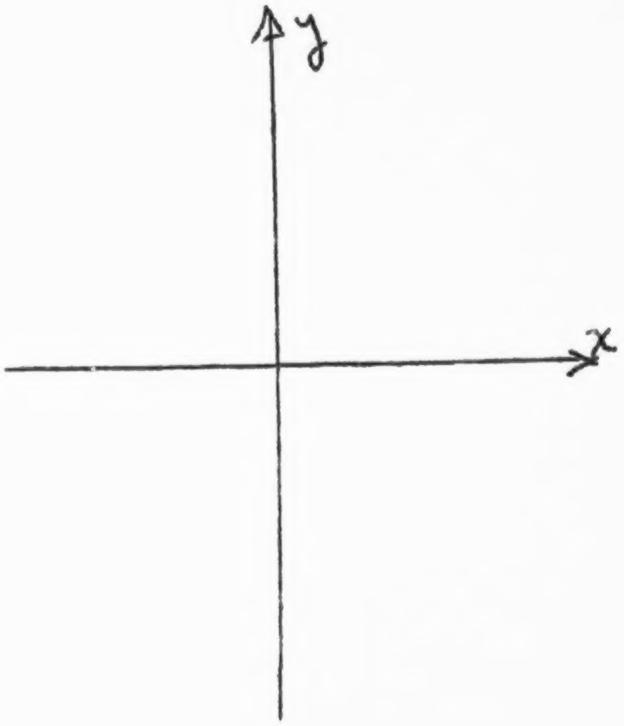
Name ______________________

Time: 10:00 ____; 11:00 ____; 12:00 ____; 1:00 ____ .

$$\begin{cases} \{(x, y) \mid y = x + 5 , x \text{ real}\} \\ \{(x, y) \mid x = -2 , x \text{ real}\} \end{cases}$$



$$\begin{cases} \{(x, y) \mid y \geq |x + 3| , x \text{ real}\} \\ \{(x, y) \mid y = 3 , x \text{ real}\} \end{cases}$$



Performance Test

Versions 2 and 3

Versions 2 and 3 contained the same problems as Version 1. The only differences between the three tests were the order in which the problems were listed.

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